MATH 307 Solutions to Midterm 2

1. Let $f(x, y) = \cos(x)\sin(x^2 + y^2)$.

$\boxed{\text{5 pts.}}$ (a) Calculate the gradient of f when $x = 0$ and $y = \frac{\sqrt{\pi}}{2}$.

Solution:

$$\nabla f(x, y) = \left\langle 2x\cos(x)\cos(x^2 + y^2) - \sin(x)\sin(x^2 + y^2), 2y\cos(x)\cos(x^2 + y^2)\right\rangle$$

At $(0, \sqrt{\pi}/2)$, this is $\left\langle 0, \frac{\sqrt{\pi}}{\sqrt{2}}\right\rangle$. (Recall that $\sin(0) = 0$, $\cos(0) = 1$, and $\cos(\pi/4) = 1/\sqrt{2}$.)

$\boxed{\text{5 pts.}}$ (b) Write the equation of the plane tangent to the surface $z = f(x, y)$ at the point $\left(0, \frac{\sqrt{\pi}}{2}, \frac{1}{\sqrt{2}}\right)$.

Solution: Recall that the plane tangent to a surface $z = f(x, y)$ satisfies the equation $z = z_0 + f_x(x - x_0) + f_y(y - y_0)$. In the current case, we have

$$z = \frac{1}{\sqrt{2}} + \frac{\sqrt{\pi}}{\sqrt{2}}(y - \frac{\sqrt{\pi}}{2}).$$

$\boxed{\text{5 pts.}}$ (c) A particle is moving along the curve $\gamma(t) = \left\langle \frac{\sqrt{\pi}}{2}\cos t, \frac{\sqrt{\pi}}{2}\sin t\right\rangle$. Find the rate of change of $f(x, y)$ along this curve when $t = \pi/2$.

Solution: This means we want to find the directional derivative in the direction of the tangent to $\gamma(t)$ at $t = \pi/2$. Notice that $\gamma(\pi/2) = \langle 0, \sqrt{\pi}/2\rangle$, and

$$\gamma'(t) = \left\langle -\frac{\sqrt{\pi}}{2}\sin t, \frac{\sqrt{\pi}}{2}\cos t\right\rangle, \quad \text{so} \quad \gamma'(\pi/2) = \left\langle -\sqrt{\pi}/2, 0\right\rangle, \quad \text{and} \quad \frac{\gamma'(\pi/2)}{|\gamma'(\pi/2)|} = \langle -1, 0\rangle.$$

Since $D_u f = (\nabla f) \cdot u$, in the present case we have

$$\left\langle 0, \frac{\sqrt{\pi}}{\sqrt{2}}\right\rangle \cdot \langle -1, 0\rangle = 0.$$

This should not be surprising: the curve is traveling parallel to the $x-$axis at this point, and from the previous part, the tangent plane is tilted only in the y direction. Thus, the value of $f(x, y)$ is not changing at this point.

15 pts. 2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x, y, z) = x^2+y^2+z^2$, $\quad g(u, v) = \begin{pmatrix} u+v \\ u^3-v \\ u-v^3 \end{pmatrix}$.

Write the derivative matrix of $f \circ g$ at a point (u, v).

Solution: Applying the chain rule, we have

$$D_g(u, v) = \begin{pmatrix} 1 & 1 \\ 3u^2 & -1 \\ 1 & -3v^2 \end{pmatrix} \quad \text{and} \quad D_f(x, y, z) = \begin{pmatrix} 2x, 2y, 2z \end{pmatrix}.$$

Thus,

$$\begin{aligned} D(f \circ g)(u, v) &= Df(u+v, u^3-v, u-v^3)Dg(u, v) \\ &= \begin{pmatrix} 2u+2v, 2u^3-2v, 2u-2v^3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3u^2 & -1 \\ 1 & -3v^2 \end{pmatrix} \\ &= \begin{pmatrix} 4u+2v+6u^5-6u^3v-2v^3, 2u+4v-2u^3-6uv^2+6v^5 \end{pmatrix} \end{aligned}$$

If you prefer, you could write the composition

$$f \circ g(u, v) = (u+v)^2 + (u^3-v)^2 + (u-v^3)^2$$

and then compute the gradient to get

$$\begin{aligned} &\left\langle 2(u+v) + 2(u^3-v)(3u^2) + 2(u-v^3), 2(u+v) - 2(u^3-v) + 2(u-v^3)(-3v^2) \right\rangle \\ &= \left\langle 4u+2v+6u^5-6u^3v-2v^3, 2u+4v-2u^3-6uv^2+6v^5 \right\rangle \end{aligned}$$

15 pts. 3. Let $f(x, y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x^2+y^2 = 0 \end{cases}$. Is $f(x, y)$ continuous at all $(x, y) \in \mathbb{R}^2$?

If not, identify any discontinuities. Justify your answer fully.

Solution: This is continuous at all points of $\mathbb{R}^2$. The only potential issue is near the origin, but it isn't hard to see that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

Perhaps the easiest way is to rewrite the function in polar coordinates.

Let $x = r\cos\theta$, $y = r\sin\theta$, so $f(x, y)$ becomes

$$\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r^2}{r^2} \cdot \frac{cos^3\theta - \sin^3\theta}{1} = r(\cos^3\theta - \sin^3\theta)$$

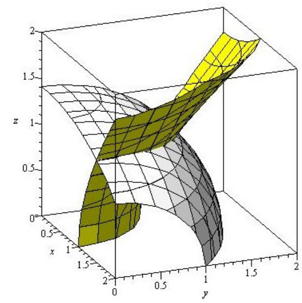
This obviously tends to 0 as $r \to 0$, no matter what θ does.

$\boxed{\text{15 pts.}}$ 4. Two surfaces are given by $f(x,y,z)=0$ and $g(x,y,z)=0$, where

$$f(x,y,z) = x^2+2y^2+3z^2-6$$
$$g(x,y,z) = x^2+y^2-z^2-1 \quad .$$

Let $\gamma(t)$ be the curve where they intersect. Determine the line tangent to γ at $(1,1,1)$.

(Note: it is not necessary to determine $\gamma(t)$, but you may.)



Solution: Implicit differentiation gives us (writing x' for dx/dt, etc.):

$$2xx' + 4yy' + 6zz' = 0 \qquad 2xx' + 2yy' - 2zz' = 0$$

Evaluating at $(1,1,1)$ and dividing by 2 gives

$$x' + 2y' + 3z' = 0 \qquad x' + y' - z' = 0$$

so $y' = -4z'$ and $x' = 5z'$. Thus, $\langle 5,-4,1\rangle$ will be tangent to the curve of intersection, and the tangent line can be written as

$$\langle 1,1,1\rangle + t\,\langle 5,-4,1\rangle$$

Alternatively, if you prefered to find $\gamma(t)$, we can do the following. You might have a minor variation that gives an equivalent answer.

First, if we set $f(x,y,z) = g(x,y,z)$, we obtain

$$y^2 + 4z^2 - 5 = 0, \qquad \text{or} \qquad y^2 = 5 - 4z^2.$$

Substituting this back into $g(x,y,z) = 0$ yields

$$x^2 + (5-4z^2) - z^2 - 1 = 0, \qquad \text{or} \qquad x^2 = 5z^2 - 4.$$

Putting these two together, and letting $z^2 = t$ gives us

$$\gamma(t) = \langle 5t-4, 5-4t, t\rangle\,, \qquad \text{with} \quad \gamma(1) = \langle 1,1,1\rangle\,.$$

Hence $\gamma'(1) = \langle 5,-4,1\rangle$. The tangent line at $t=1$ can be written as $\langle 1,1,1\rangle + t\,\langle 5,-4,1\rangle$, just as via implicit differentiation.

Note that we cannot just take the partials of $f(x,y,z) - g(x,y,z)$ and plug in $(1,1,1)$; a few people tried this.

You could, however, observe that the tangent line to $\gamma(t)$ lies in both tangent planes to f and g. Thus, you could find the normals $\nabla f(1,1,1) = \langle 2,4,6\rangle$ and $\nabla g(1,1,1) = \langle 2,2,-2\rangle$. Then their cross product is $\langle 20,-16,4\rangle$, and so the tangent line can be written as $\langle 1,1,1\rangle + s\,\langle 20,-16,4\rangle$.

$\boxed{\text{15 pts.}}$ 5. Find the point on the sphere $x^2 + y^2 + z^2 = 1$ which is furthest from the point $(1, 2, 3)$.

Solution: The easiest way to do this problem is to notice that the solution must lie on the line $\langle t, 2t, 3t\rangle$. We also require $x^2 + y^2 + z^2 = 1$, so the answer must be $\left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$.

If you want to work harder, you could use Lagrange multipliers. We want to maximize $f(x, y, z) = (x-1)^2+(y-2)^2+(z-3)^2$ subject to the constraint $g(x, y, z) = x^2+y^2+z^2-1 = 0$. Hence, we find the solutions to $\nabla f + \lambda\nabla g = 0$. We have

$$2(x-1) + 2\lambda x = 0 \qquad 2(y-2) + 2\lambda y = 0 \qquad 2(z-3) + 2\lambda z = 0$$

so $x = \frac{1}{1-\lambda}$, $y = \frac{2}{1-\lambda}$, $z = \frac{3}{1-\lambda}$. Since we need $x^2 + y^2 + z^2 = 1$, we get $\left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$ as the maximum.

$\boxed{\text{15 pts.}}$ 6. Find all of the critical points of $x^3 - 6xy - 6y^2$. For each, state whether it is a local minimum, local maximum, or neither.

Solution: We calculate the gradient as $\langle 3x^2 - 6y, -6x - 12y\rangle$. Thus, we must have

$$3x^2 - 6y = 0 \qquad -6x - 12y = 0$$

or $x^2 = 2y$, $x = -2y$. Hence $4y^2 - 2y = 0$, and so $y = 0$ or $y = 1/2$. This means the only critical points are $(0, 0)$ and $(-1, 1/2)$.

Note that $f_{xx}(x, y) = 6x$, $f_{yy}(x, y) = -12$, and $f_{xy}(x, y) = -6$.
At $(0, 0)$, the discriminant $f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$, so this is a saddle point.
At $(-1, 1/2)$, $f_{xx}f_{yy} - f_{xy}^2 = +36$, and both f_{xx} and f_{yy} are negative. Thus $(-1, 1/2)$ is a local maximum.

A picture of the surface is at right. Note that the x-axis increases to the left in the picture. (sorry, it is too hard to see otherwise.)