MATH 200, Lec 2 Solutions to Midterm 2

1. $\boxed{\text{10 points}}$ Prove that there is no rational number whose square is 3.
   You may assume that if a is an integer, a^2 is divisible by 3 if and only if a is divisible by 3.

   Solution: Suppose there was a rational number x whose square was divisible by 3. Then there would be integers p and q with no common divisors so that $x = p/q$ and $x^2 = 3$.

   Thus
   $$\frac{p^2}{q^2} = 3, \quad \text{and so} \quad p^2 = 3q^2$$
   which means p is divisible by 3, that is, there is an integer a so that $p = 3a$. Hence
   $$3q^2 = p^2 = (3a)^2 = 9a^2,$$
   and so $q^2 = 3a^2$. This means q is also divisible by 3, which contradicts our assumption that p and q had no common divisors.

2. (a) $\boxed{\text{5 points}}$ Show that if A and B are disjoint denumerable sets, then $A \cup B$ is also denumerable.

   Solution: Since A and B are denumerable, we have
   $$A = \{a_1, a_2, a_3, \dots\} \qquad B = \{b_1, b_2, b_3, \dots\},$$
   that is, we have bijections $f : \mathbb{Z}^+ \to A$ and $g : \mathbb{Z}^+ \to B$. What we need is to give a way to list $A \cup B$, that is, a bijection $h : \mathbb{Z}^+ \to A \cup B$.
   Note that we can't just list the elements of A followed by those of B: since A is infinite, we'll never get to B. So we take the "one for you, one for me" strategy, and alternate between the two sets, that is,
   $$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}.$$
   More formally, we can write the bijection $h : \mathbb{Z}^+ \to A \cup B$ as
   $$h(i) = \begin{cases} f\left(\frac{i+1}{2}\right) & \text{if } i \text{ is odd} \\ g\left(\frac{i}{2}\right) & \text{if } i \text{ is even} \end{cases}$$

(b) $\boxed{\text{5 points}}$ Show that if X is an uncountable set and $A \subseteq X$ is denumerable, then the complement of A in X (that is, $X - A$) must be uncountable.
You may use the first part of this question, even if you couldn't do it

Solution: We can do this by contradiction. If $X - A$ is not uncountable, then it must be countable, that is either finite or denumerable.

If $X - A$ is denumerable, we have X expressed as the union of two denumerable sets: $X = A \cup (X - A)$, and so by the first part of the problem, X is denumerable, giving a contradiction.

Similarly, if $X - A$ is finite, since A is denumerable, their union is again denumerable, giving a contradition. (There is a theorem in the text to this effect. However, the proof is simple: If $|X - A| = n$, then we can write $X - A = \{x_1, x_2, x_3, \ldots, x_n\}$, and so $X = \{x_1, x_2, x_3, \ldots, x_n, a_1, a_2, a_3, \ldots\}$.)

3. Three people decide to get tacos, and the tacqueria serves five kinds of tacos: beef, chicken, pork, fish, and vegetarian. Each person orders exactly one taco.

(a) $\boxed{\text{5 points}}$ How many choices are possible if we record who selected which dish (as the waiter should)?

Solution: Each person can choose one of five types of taco, so there are $5 \cdot 5 \cdot 5 = 5^3 = 125$ possible choices for all three.

(b) $\boxed{\text{5 points}}$ How many choices are possible if we forget who ordered which dish (as the chef might)?
Be careful, this is more complicated than it may seem at first.

Solution: Here there is a slight complication since more than one person might order the same type of taco. We just count the three cases separately.

- First, if all three get the same type of taco, there are 5 possibilities.
- If two get the same type of taco, and one gets something else, we have 5 choices for the two that are the same, and 4 choices remain for the different one. This gives us 20 possibilities.
- Finally, if all three get different types, this means we have $\binom{5}{3} = 10$ possibilities.

Altogether, this gives us $5 + 20 + 10 = 35$ different orders from the chef's point of view.

4. $\boxed{\text{5 points}}$ What is the coefficent of x^9 in the expansion of $(x+2)^{12}$?

Solution: We apply the binomial theorem, which tells us that the term involving x^9 looks like

$$\binom{12}{9} x^9 2^3 = 8\frac{12!}{9!3!}x^9 = 8 \cdot 220 x^9 = 1760x^9$$

so the coefficient of x^9 is 1760.

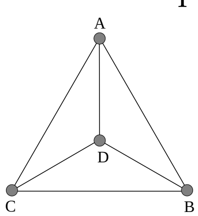
5. $\boxed{\textbf{10 points}}$ **Using only the definitions and axioms** on the back of the cover sheet, prove that if ℓ, m, and n are lines so that ℓ is parallel to m, and m is parallel to n, then ℓ is parallel to n.

Solution: If $\ell = m$, the result follows immediately, since $\ell \parallel m$.

Now suppose $\ell \neq m$, so ℓ and m have no points in common. Either ℓ and n have no points in common (in which case we are done, since then they are parallel), or they share at least one point. Call this point P. Since ℓ and m are disjoint, P is not on m, and so by the parallel axiom there is a unique line which is parallel to m and passes through P. Since both ℓ and n are parallel to m and pass through P, the only possibility is that they are equal. By the definition of parallel, if $\ell = n$, then also $\ell / parallel n$, as desired.

(You can also do this second part by contradiction. The argument is much the same.)

6. For each of the interpretations of the terms point, line, and distance given below, determine if they are consistent with the axioms given on the back of the cover sheet. If the interpretation is not consistent, state **all** axioms it contradicts, and explain why.



(a) $\boxed{\text{5 points}}$ The plane contains exactly four points, A, B, C, and D. There are six lines: $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{CD}$, and the distances between points are given by $|AD| = |BD| = |CD| = 1$ and $|AB| = |BC| = |CA| = \sqrt{3}$.

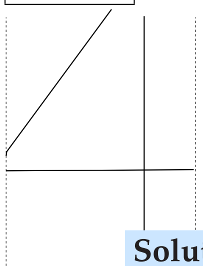
Solution: We'll check each of the axioms in turn:

Incidence Axiom: Satisfied. Each line contains two points, each pair of points lies on a unique line, and each line has at least one point not on it.

Parallel Axiom: Satified. For each line, there is another line which is disjoint from it, and hence parallel. Specifically, $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$, $\overleftrightarrow{AC} \| \overleftrightarrow{BD}$, and $\overleftrightarrow{AD} \| \overleftrightarrow{BC}$.

Ruler Axiom: This one fails. Each line has only two points, and $\mathbb{R}$ is uncountable. So there is no possibility of a bijection of any of the lines with $\mathbb{R}$.

(b) $\boxed{\text{5 points}}$ Points are elements $(x, y) \in \mathbb{R}^2$ with $-1 < x < 1$. A line is the set of points which satisfy $y = mx+b$ where m and b are real numbers (and $-1 < x < 1$); in addition, the points which satisfy $x = a$ where $-1 < a < 1$ are also lines. The distance between two points (x_1, y_1) and (x_2, y_2) is given by $\sqrt{\left(\frac{x_1}{x_1^2-1} - \frac{x_2}{x_2^2-1}\right)^2 + (y_1-y_2)^2}$



Solution:

Incidence Axiom: As before, the incidence axiom holds. Given any two points (x_1, y_1) and (x_2, y_2) in the strip, we can find a unique line passing through them as follows: If $x_1 = x_2$, then the line is $x = x_1$. Otherwise, the line has the equation

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

Each line contains infinitely many points, and for each line, there are plenty of points not on it.

Parallel Axiom: This one fails. Here is a counterexample: Take the line $y = 0$, and the point $(0, 4)$. Then any line of the form $y = mx+4$ with $-4 < m < 4$ will pass through $(0, 4)$ and be disjoint from $y = 0$ (and hence parallel to it).

Ruler Axiom: The ruler axiom holds. The easy way to see this is to notice that the given distance formula "stretches horizontal distances" near the sets $\{(x, y) \mid x = \pm 1\}$. The transformation

$$f : \{(x, y) \in R^2 \mid -1 < x < 1\} \to \mathbb{R}^2 \text{ given by } f(x, y) = \left(\frac{x}{1-x^2}, y\right)$$

is a bijection of our strip with the regular plane $\mathbb{R}^2$ sending vertical lines to vertical lines, and a line segment of the form $y = mx + b$ $(-1 < x < 1)$ to a curve with horizontal asymptotes at $y = m + b$ and $y = -m + b$. The distance formula given just measures the distance (in $\mathbb{R}^2$) between points on this curve.