First Exam MAT 200, Fall 2002
Solutions to yellow exam

1. Consider the statement If I don't study, then I won't pass this test.
Which of the following is logically equivalent to this statement?

a. Either I don't study or I'll pass this test.

b. If I study, then I will pass this test.

c. Either I study or I won't pass this test.

d. I will study and I will pass this test.

e. If I don't pass this test, then I didn't study.

f. This test is too hard.

Solution: The correct statement is

Either I study or I won't pass this test.

One way to see this is to use symbolic notation. If we let P be the statement "I don't study" and Q be the statement "I won't pass this test", then the original question is of the form $P \Rightarrow Q$, which is equivalent to $\sim P \vee Q$. Since "I will study" is the negation of "I don't study", we have the result.

None of the other choices are equivalent; symbolically, each of the others is one of the following: $P \vee \sim Q$, $\sim P \Rightarrow \sim Q$, $\sim P \wedge \sim Q$, $Q \Rightarrow P$, or "This test is too hard".

2. For each of the following, indicate whether it is always true, sometimes true, or never true. A, B, C are used to denote sets, P, Q, R are used for propositional variables. $\mathcal{P}(A)$ is the power set of A.

a. $(A - B) \cap (B - A) = \emptyset$ Always

Solution: This is always true, since if $x \in A - B$, then $x \notin B$ so $x \notin B - A$. Since $A - B$ and $B - A$ have no elements in common, their intersection is empty.

b. $\sim(P \wedge Q) \Rightarrow (\sim P \wedge \sim Q)$ Sometimes

Solution: While this isn't generally true, it is true sometimes, for example if $P = Q$.

c. $\forall x \in \mathbb{N}\ \exists y \in \mathbb{R}\ y^2 = x$ Always

Solution: This says for every natural number x, there is a real number y which is a square root of x. Since the natural numbers are all positive, this is always true.

d. $\exists x \in \mathbb{N}\ \forall y \in \mathbb{R}\ y^2 = x$ Never

Solution: This is never true; it says that there is a universal natural number x so that if we square any real number y, we get x.

e. For any set B, $A - B \in \mathcal{P}(A)$ Always

Solution: For any B, $A - B \subseteq A$, since $A - B$ is just A with any elements common to B removed. Since it is a subset of A, it is an element of the power set of A.

f. $A \subseteq B \Leftrightarrow (A - B) = \emptyset$ Always

Solution: First, suppose $A \subseteq B$, and let x be any element of A. Then $x \in B$, so $x \notin A - B$. Since $A - B \subseteq A$, $A - B$ can have no elements, so it is the empty set.
Now, suppose $A - B = \emptyset$, and let y be any element of A. Since $A - B$ has no elements, y must also be an element of B. This means that $A \subseteq B$.

g. $(P \Rightarrow (Q \wedge \sim Q)) \Leftrightarrow \sim P$ Always

Solution: This is a standard tautology, which is the basis of proof by contradiction.

h. $\exists n \in \mathbb{N}\ m = n^2$ Sometimes

Solution: Since m is not quantified, this is a sentence $P(m)$, which says that given m, there is some natural number n which is a square root of m. Since $P(25)$ is true but $P(2)$ is false, this is only sometimes true. Also, it is important to note that nowhere is the domain of m mentioned... what if m was a sphere? Then P makes no sense at all.

3. Using the axioms and theorems on the previous page of this exam, as well as any tautologies, give explicit reasons for each step of the formal proof of

$$\forall x\ x > 0 \Rightarrow x^{-1} > 0$$

1. Suppose $x > 0$	Hypothesis
2. Assume $\sim(x^{-1} > 0)$	Assumption for indirect proof
3. $x^{-1} > 0 \lor x^{-1} < 0 \lor x^{-1} = 0$	Trichotomy (Ax. 19)
4. $x^{-1} < 0 \lor x^{-1} = 0$	steps 2+3, and tautology $P \lor (Q \land \sim Q) \Rightarrow P$
5. Case 1: $x^{-1} = 0$	Assumption for proof by cases
6. $x \cdot x^{-1} = 1$	existence of mult. inverse (Ax. 15)
7. $x \cdot x^{-1} = 0$	Thm 3 applied to step 5
8. $0 = 1$	steps 6, 7 and transitivity of $=$ (axiom 3)
9. $0 \neq 1$	axiom 16
10. Contradiction.	$\sim(P \land \sim P)$, last two steps.
11. Case 2: $x^{-1} < 0$	Assumption for proof by cases
12. $x \cdot x^{-1} < x \cdot 0$	Steps 1, 11, and axiom 21
13. $x \cdot x^{-1} = 1$	existence of mult. invers (axiom 15)
14. $x \cdot 0 = 0$	Thm. 3
15. $1 < 0$	from steps 12, 13, 14, substitution
16. $1 > 0$	Thm. 4
17. Contradiction.	$\sim(P \land \sim P)$, last two steps.
18. $x^{-1} > 0$	Conclusion of indirect proof with assumption 2.
19. $x > 0 \Rightarrow x^{-1} > 0$	Result of conditional proof, 1 & 18.
20. $\forall x\ x > 0 \Rightarrow x^{-1} > 0$	Universal Generalization (1,19)

4. An infinite sequence of real numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ is said to converge to L if for every number $\epsilon > 0$, there is a natural number N so that for any $k > N$, we must have $|a_k - L| < \epsilon$. Write a statement, either symbolically or in English, that expresses "the sequence a_n does not converge to L". **This statement must not use the connective $\sim$ or the word "not".** You may use the fact that $\sim(a < b)$ can be written as $a \geq b$. You'll probably find it helpful to write a symbolic version of the original statement first.

Solution: First, we translate this to symbols, which almost goes directly:
"for every number $\epsilon > 0$" is written as $\forall \epsilon > 0$, "there is a natural number N" becomes $\exists N \in \mathbb{N}$,
"for any $k > N$" is $\forall k > N$, and "we must have $|a_k - L| < \epsilon$" is just $|a_k - L| < \epsilon$.
Putting it all together, we get

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall k > N \;\; |a_k - L| < \epsilon$$

We are asked to negate this statement and write the result without using the connective $\sim$, so first let's put the connective in, and get rid of it step by step.

$$\begin{gathered}
\sim(\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall k > N \;\; |a_k - L| < \epsilon) \\
\exists \epsilon > 0 \;\; \sim(\exists N \in \mathbb{N} \; \forall k > N \; |a_k - L| < \epsilon) \\
\exists \epsilon > 0 \; \forall N \in \mathbb{N} \;\; \sim(\forall k > N \; |a_k - L| < \epsilon) \\
\exists \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists k > N \;\; \sim(|a_k - L| < \epsilon) \\
\exists \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists k > N \;\; |a_k - L| \geq \epsilon
\end{gathered}$$

We can say this in words as

> There is an $\epsilon > 0$ so that for every natural number N there is a $k > N$ for which $|a_k - L| \geq \epsilon$.

5. Using mathematical induction, prove that for any natural number n,

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

Solution: For every $n \in \mathbb{N}$, we want to establish the statement
$P(n): \quad 1 + 3 + 5 + \ldots + (2n - 1) = n^2$ using induction.

First we show our base case ($P(1)$): This is $1 = 1^2$, which is certainly true.

Now for the induction step, we derive $P(n + 1)$ assuming $P(n)$. That is, we want to show that

$$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2$$

assuming $P(n)$ holds. Note that it is important not to assume the equality above; we should start with one side and derive the other. Thus:

$$\begin{array}{rrcl}
1 + 3 + 5 + \ldots + (2n - 1) & +(2n + 1) & = & \\
n^2 & +(2n + 1) & = & \text{(using } P(n)\text{)} \\
(n + 1)^2 & & & \text{by factoring}
\end{array}$$

Since this is exactly $P(n + 1)$, we have shown $P(n) \Rightarrow P(n + 1)$, which completes the proof.

6. In the problem below, use the following predicates and constants, as well as the usual sets $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N}$.

$P(x)$ means x is a person. $T(t)$ means t is a time.
$F(x,t)$ means x can be fooled at time t.
$M(x)$ means x is a man. $S(x,y)$ means x shaves y.
b is the barber.

Write each of the statements below symbolically, using only the predicates, constants and sets above, the connectives $\sim$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, the binary relations $=$ and $>$, the quantifiers $\forall$ and $\exists$, parenthesis, and any necessary variables.

a. Any rational number can be expressed as the ratio of two integers.

Solution: This says that for every rational number, there are two integers whose ratio equals that rational. That is

$$\forall r \in \mathbb{Q}\ \exists x, y \in \mathbb{Z}\ \ r = \frac{x}{y}$$

b. You can fool some of the people all of the time, and you can fool all of the people some of the time, but you can't fool all of the people all of the time.
You may assume that "You can fool Mr. Jones" is the same as "Mr. Jones can be fooled".

Solution: This one is a bit trickier. To make it easier, we could say that the domain of the variable t is times, and the domain of the variable x is people.

Then "you can fool some of the people all of the time" could be said as "there are people who can be fooled at all times", and "you can fool all of the people some of the time" would mean that for any person, there is a time when s/he can be fooled. Thus, we get

$$\exists x\ \forall t\ F(x,t) \wedge \forall x\ \exists t\ F(x,t) \wedge\ \sim(\forall x\ \forall t\ F(x,t))$$

If we didn't state a domain in advance, we have to work harder for the same effect:

$$(\exists x\ P(x) \wedge \forall t\ T(t) \Rightarrow F(x,t)) \wedge (\forall x\ P(x) \Rightarrow \exists t\ (T(t) \wedge F(x,t))) \quad \wedge$$
$$\sim(\forall x\ \forall t\ (P(x) \wedge T(t)) \Rightarrow F(x,t))$$

c. The barber shaves every man who does not shave himself.

Solution: If a man does not shave himself, then the barber shaves him. So:

$$\forall x\ M(x) \wedge\ \sim S(x,x) \Rightarrow S(b,x)$$

d. Between any two real numbers, there is a rational number.

Solution: This says, if x and y are any two real numbers, then there is a rational number r which is between them. We can either assume that $x < y$, or handle both possiblities:

$$\forall x, y \in \mathbb{R}\ \exists r \in \mathbb{Q}\ (x < r < y \vee y < r < x)$$

or

$$\forall x, y \in \mathbb{R}\ x < y \Rightarrow \exists r \in \mathbb{Q}\ (x < r < y)$$

Extra Credit. In this problem, you are to place an X in either the box on the left or the box on the right, **but not both**. Putting an X in a "good" box is worth 3 points, but putting an X in a "bad" box is worth nothing. It may be that I was very kind and both boxes are good, or I may be feeling nasty and neither box is good. If you put an X in both boxes, I will take away 10 points, even if both boxes are good. So don't do that.

The statement above the left box is true if and only if the left box is good.

The statement above the right box is true if and only if the right box is bad.

It doesn't matter which box you choose	You should choose the other box
good $\Leftrightarrow$ true	good $\Leftrightarrow$ false
	X

For 7 additional points, you must say which boxes are good, which boxes are bad, and give an informal but complete proof of your claim.

Solution: Let's call the box labeled "You should choose the other box" box O, and the one that says "It doesn't matter which box you choose" box M.

We need to consider the four possible cases:

Box M bad, Box O bad: Since box M is bad "It doesn't matter which box you choose" is supposedly false. Thus, it does matter. But if both boxes are bad, it doesn't matter, so this is a contradiction.

Box M bad, Box O good: Again, if box M is bad, it matters which box we choose. If Box O is good, then "You should choose the other box" is false, so we should not choose box M. Thus, choosing box O is what we want.

Box M good, Box O bad: If box M is good, then it doesn't matter which box we choose. But if Box O is bad, we won't get any points by choosing Box O, so it *does* matter. This is a contradiction.

Box M good, Box O good: If box M is good, then it doesn't matter which box we choose. Since both boxes are good, this is OK so far. Now, if Box O is good, then the statement "You should choose the other box" is false, so there is no gain to be had from choosing Box M. This is not a contradiction, since if both boxes are good, we can choose either one.

From the above reasoning, the following is a true statement:

$$(\text{Box M bad} \wedge \text{Box O good}) \vee (\text{Box M good} \wedge \text{Box O good})$$

No matter what, Box O must be good, so we should pick that box.

Indeed, Box 0 was the good box.