3. The parallel axiom

Axiom 8 (Parallel Axiom)**.** *Given a line k, and a point A not on k, there is exactly one line m passing through A and parallel to k.*

We remark that the point of the axiom is not the existence of the parallel, but the uniqueness. We will see below that existence actually follows from what we already know.

It is sometimes convenient to think of a line as being parallel to itself, so we make the following formal definition. Two lines are not parallel if they have exactly one point in common; otherwise they are parallel.

Theorem 3.1. *In the set of all lines in the plane, the relation of being parallel is an equivalence relation.*

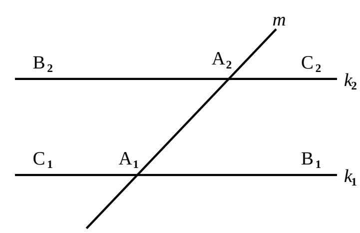
Proof. First, since a line has infinitely many points in common with itself, it is parallel to itself; hence the relation is reflexive (this is the point of the strange definition).

Second, the definition is obviously symmetric; it is defined in terms of the two lines; not one with relation to the other.

Third, suppose k is parallel to m, and m is parallel to n. There is obviously nothing further to prove unless the three lines are distinct. Assume that k and n are not parallel. Since two lines are either equal, parallel, or have exactly one point in common, we must have that k and n have a point in common. But this contradicts Axiom 8. □

3.1. Alternate interior angles. We will meet the following situation some number of times. We are given two lines k_1 and k_2, and a third line m, where m crosses k_1 at A_1 and m crosses k_2 at A_2. Choose a point $B_1 \neq A_1$ on k_1, and choose a point $B_2 \neq A_2$ on k_2, where B_1 and B_2 lie on opposite sides of the line m. Then $\angle B_1A_1A_2$ and $\angle B_2A_2A_1$ are referred to as alternate interior angles.

In any given situation, there are two distinct pairs of alternate interior angles. That is, let C_1 be some point on k_1, where B_1 and C_1 lie on opposite sides of m, and let C_2 be some point on k_2, where C_2 and B_2 lie on opposite sides of m. Then one could also regard $\angle C_1A_1A_2$ and $\angle C_2A_2A_1$ as being alternate interior angles. However, observe that $m\angle B_1A_1A_2 + m\angle C_1A_1A_2 = \pi$ and $m\angle B_2A_2A_1 + m\angle C_2A_2A_1 = \pi$. It follows that one pair of alternate interior angles are equal if and only if the other pair of alternate interior angles are equal.



Proposition 3.2. *If the alternate interior angles are equal, then the lines k_1 and k_2 are parallel.*

Proof. Suppose not. Then the lines k_1 and k_2 meet at some point D. If necessary, we interchange the roles of the B_i and the C_i so that $\angle B_1A_1A_2$ is an exterior angle of $\triangle A_1A_2D$. Then D and B_2 lie on the same side of m, so $\angle DA_2A_1 = \angle B_2A_2A_1$. By the exterior angle inequality,

$$m\angle B_1A_1A_2 > m\angle A_1A_2D = m\angle B_2A_2A_1 = m\angle B_1A_1A_2,$$

so we have reached a contradiction. □

3.2. EXISTENCE OF PARALLEL LINES. Let k_1 be a line, and let A_2 be a point not on k_1. Pick some point A_1 on k_1 and draw the line m through A_1 and A_2. By Axiom 7, we can find a line k_2 through A_2 so that the alternate interior angles are equal. Hence we can find a line through A_2 parallel to k_1.

Theorem 3.3 (alternate interior angles equal)**.** *Two lines k_1 and k_2 are parallel if and only if the alternate interior angles are equal.*

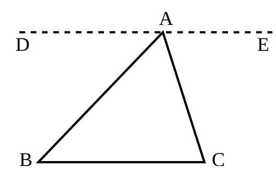
Proof. To prove the forward direction, construct the line k_3 through A_2, where there is a point B_3 on k_3, with B_3 and B_2 on the same side of m, so that $m\angle B_3A_2A_1 = m\angle B_1A_1A_2$. Then, by Prop. 3.2, k_3 is a line through A_2 parallel to k_1. Axiom 8 implies $k_3 = k_1$. Hence $m\angle B_3A_2A_1 = m\angle B_2A_2A_1$, and the desired conclusion follows.

The other direction is just Prop. 3.2, restated as part of this theorem for convenience. □

3.3. THE SUM OF THE ANGLES OF A TRIANGLE.

Theorem 3.4. *The sum of the measures of the angles of a triangle is equal to π.*

Proof. Consider $\triangle ABC$, and let m be the line passing through A and parallel to BC. Let D and E be two points on m, on opposite sides of A, where D and C lie on opposite sides of the line AB. Then B and E lie on opposite sides of AC.



Exercise 3.1: Use alternate interior angles to complete the proof of this theorem.

□

A quadrilateral is a region bounded by four line segments, so it has four verticies on its boundary.

Corollary 3.5. *The sum of the measures of the angles of a quadrilateral is 2π.*

Proof. Cut the triangle into two triangles, and do the obvious computation. □

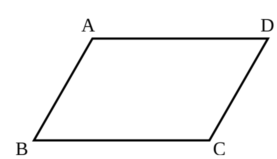
A rectangle is a quadrilateral in which all four angles are right angles.

Theorem 3.6. *If $ABCD$ is a rectangle, then AB is parallel to CD, and $|AB| = |CD|$. Similarly, BC is parallel to AD and $|BC| = |AD|$.*

Exercise 3.2: Prove this theorem.

i. Prove that opposite pairs of sides are parallel.
ii. Now cut the rectangle into two triangles; prove that these two triangles are congruent. Conclude that opposite sides of the rectangle have equal length.

Somewhat more generally, a parallelogram is a quadrilateral $ABCD$ in which opposite sides are parallel; that is, AB is parallel to CD, and AD is parallel to BC.
A rectangle with all four sides of equal length is a square; a parallelogram with all four sides of equal length is a rhombus.



Theorem 3.7. *Let $ABCD$ be a parallelogram. Then $m\angle A = m\angle C$; $m\angle B = m\angle D$; $|AB| = |CD|$; and $|BC| = |AD|$.*

Exercise 3.3: Prove this theorem. (Hint: Draw a diagonal.)

Theorem 3.8. *If $ABCD$ is a quadrilateral in which $|AB| = |CD|$ and $|AD| = |BC|$, then $ABCD$ is a parallelogram.*

Exercise 3.4: Prove this theorem.

Theorem 3.9. *Let $ABCD$ be a parallelogram with diagonals of equal length (that is, $|AC| = |BD|$). Then $ABCD$ is a rectangle.*

Exercise 3.5: Prove this theorem.