MAT 200
COURSE NOTES ON GEOMETRY

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Contents

1. Introduction

1.1. Physical vs. Ideal. What is a triangle? Is a triangle a physical object made up of 3 straight pieces of wood or metal or somesuch, joined at the corners, or is it an ideal object consisting of lines that have no width lying in a plane that has no thickness?

1.2. The idea of constructibility. Historically, all lengths and angles are somehow constructible. That is, they are abstract objects that, in some sense are capable of being realized as physical objects. We will take a somewhat different point of view; we will assume familiarity with real numbers, and with the correspondence between real numbers and points on a line.

1.3. Basic objects. The plane, lines, points, length and distance, angle measure.

1.4. Basic concepts.

- Points lie on lines.
- If two distinct points lie on a line, then the length of the line segment between these points is well defined.
- A point separates a line into two half-lines.
- Two distinct points on the same line separate it into a line segment, which has a length (the length is a positive number), and two half-lines.
- A line separates the plane into two half-planes, which are regions (in modern terms, a region is a connected open subset of the plane).
- Two distinct lines either meet at a point, or are disjoint, in which case they are parallel.

1.5. Basic axioms. These are the first few; a few more will follow. The reader should be aware that the numbering of the axioms, as well as the theorems, propositions, etc. is unique to these notes. It is also possible to have the same notion of planar geometry with a slightly different collection of axioms, but these are what we shall use.

Axiom 1. *Two distinct lines intersect in at most one point.*

Axiom 2. *Any two distinct points lie on a line.*

Axiom 3. *If two lines intersect in a point, they separate the plane into 4 regions, called sectors, and they define an angle in each of these sectors; the sum of the measures of any two adjacent angles is π.*

Axiom 4. *If two lines do not intersect, they divide the plane into three regions, with exactly one of them, the one between the two lines, having both lines on its boundary.*

Exercise 1.1: Using the above axioms, show that given any two distinct points, there is exactly one line that contains them both.

1.6. Basic notations. If A and B are distinct points, then the (unique) line on which they lie is denoted by AB. The line segment between A and B is also denoted by AB; this should cause no confusion. The length of the line segment AB is denoted by $|AB|$.

If AB and AC are distinct lines or line segments, the angle between them is denoted by $\angle BAC$, and its measure is denoted by $m\angle BAC$. We may use $\angle A$ to denote an angle which has the point A at its vertex, if it is clear from the context which angle is being referred to.

It is also not unusual to use Greek letters such as α, β, θ, φ, etc. to denote both angles and their measures.

Theorem 1.1 (Vertical angles)**.** *Vertical angles are equal. That is, if two distinct lines intersect at a point, the measure of the angles of any two non-adjacent sectors is equal.*

Exercise 1.2: Prove Thm. 1.1. You may use any of the axioms above, along with logical axioms and results for real numbers (since angle measure is a real number).

1.7. Basic constructions. (more will follow)

- Any line segment can be extended in either direction, or in both directions.
- If $|AB| < |CD|$, then there is a point E on the line CD, where E lies between C and D, so that $|AB| = |CE|$ (see also Axiom **??**).
- If A and B are points on the line k, and we are given an angle $\angle CDE$, where $m\angle CDE \neq \pi$, then we can construct points F and G, one on each side of the line $k = AB$, so that $m\angle BAF = m\angle BAG = m\angle CDE$. (see also Axiom **??**).

2. Triangles and congruence of triangles

2.1. Basic measurements. Three distinct lines, a, b and c, no two of which are parallel, form a triangle. That is, they divide the plane into some number of regions; exactly one of them, the triangle, is bounded, and has segments of all three lines on its boundary.

The triangle with vertices A, B, C is denoted by $\triangle ABC$, where A is the point of intersection of the lines b and c; B is the point of intersection of the lines a and c; and C is the point of intersection of the lines a and b. These points of intersection divide each of the lines into two unbounded half-lines and one bounded line segment, called a side of the triangle.

The triangle, $\triangle ABC$, defines 6 numbers, the angle measures (also called the angles) at the vertices A, B and C, and the lengths of the sides, which are the line segments BC, AC and AB.

The angle measure at for example the vertex A is denoted by $m\angle A$, or $m\angle BAC$.

2.2. Historical note. The use of the phrase "measure of an angle" is relatively modern. Up to about 50 years ago, the measure of the angle at A was simply denoted by A or $\angle A$, and it was left to the reader to distinguish between the angle and its measure. When convenient, we will follow this convention, and use the same notation for an angle and its measure.

2.3. More on measurements. We will always give angle measures in radians, so, if A B and C all lie on a line, with B between A and C, then $m\angle ABC = \pi$.

We denote the length of the side AB, for example, by $|AB|$. Until modern times, the side and its length were denoted by the same symbol, and the reader had to figure out which is which from the context. As with angles, when convenient, we will also use the same notation for a line segment and its length.

The pair of lines, a and b, for example, determines two angles; the question of which of these angles is determined by the triangle can be stated in words with difficulty; we will leave this as visually obvious.

2.4. Congruence. Two triangles, $\triangle ABC$ and $\triangle A'B'C'$, are congruent if the corresponding angles have equal measures, and the corresponding sides have equal lengths. That is, the triangles, $\triangle ABC$ and $\triangle A'B'C'$ are congruent if $m\angle A = m\angle A'$; $m\angle B = m\angle B'$; $m\angle C = m\angle C'$; $|AB| = |A'B'|$; $|AC| = |A'C'|$; and $|BC| = |B'C'|$. In this case, we write $\triangle ABC \cong \triangle A'B'C'$.

For physical triangles, two triangles are congruent if they exactly match if you put one on top of the other. Another way of saying this, for ideal triangles, is that there is an isometry of the plane (a composition of rotation, translation and reflection) that maps one exactly onto the other.

Exercise 2.1: Show that congruence of triangles is an equivalence relation.

2.5. Important remark about notation. It is essentially obvious that congruence of triangles is an equivalence relation. However, the statement that $\triangle ABC \cong \triangle A'B'C'$ says nothing about whether $\triangle BCA$ is or is not congruent to $\triangle A'B'C'$. More precisely, the statement $\triangle ABC \cong \triangle A'B'C'$ not only tells you that these two triangles are congruent, but also tells you that $m\angle A = m\angle A'$, $|AB| = |A'B'|$, etc.

2.6. The axiom for congruence.

Axiom 5 (ASA)**.** *If* $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ *and* $|AB| = |A'B'|$, *then* $\triangle ABC \cong \triangle A'B'C'$.

It is common to refer to the above angle as "Angle-Side-Angle" or ASA.

For physical triangles this is essentially obvious. If you know the length of a side, and you know the two angles, then the lines on which the other sides lie are determined, so the third vertex is also determined.

2.7. Exercises. A physical triangle is determined by 6 pieces of information, the 3 lengths and the 3 angles. There are 6 possible statements concerning 3 pieces of information. Convince yourself that AAA and SSA are false, while AAS, ASA, SAS and SSS are true. (There is nothing here for you to hand in, but you need this information for the next two questions.)
Remark: One of these, AAS, is not obvious; in fact it is false in spherical geometry.

In the following few exercises, when you are asked to prove something you may assume that AAS, ASA, SAS and SSS are true. One other fact that you may use is Thm. 3.4: the sum of the angles of a triangle is π. Note that *this is only for these exercises*; in general we cannot assume things we have not proven or taken as an axiom, because we may wind up applying circular reasoning (that is, giving a proof that something is true which implicitly assumes it was true to begin with.) But the main point of this exercise is to get you thinking about how geometry works, so we can relax our restrictions a little.

Exercise 2.2: Is it true that no 2 pieces of information suffice to determine a triangle? That is, can you find two pieces of information so that if you have any two triangles for which these two measurements are the same, the triangles must necessarily be congruent. Prove your answer.

Exercise 2.3: What about 4 pieces of information; i.e., do any four pieces of information suffice for congruence of triangles? Prove your answer.

A quadrilateral is a region bounded by four line segments; that is, it is a four-sided figure. The quadrilateral with vertices, A, B, C and D, in this order, is determined by the four line segments connecting A and B, B and C, C and D, and connecting D and A. For $ABCD$ to form a quadrilateral, these segments must not intersect except at the verticies.

A quadrilateral defines 8 pieces of information: the lengths of the four sides, and the measures of the four angles. Two quadrilaterals are congruent if these 8 pieces of information agree.

What is the minimal number of pieces of information one needs about two quadrilaterals to prove that they are congruent? (No response needed here, but you need the answer for the next question.)

Exercise 2.4: State and prove one congruence theorem for quadrilaterals, where the hypothesis consists of the minimal number of pieces of information.

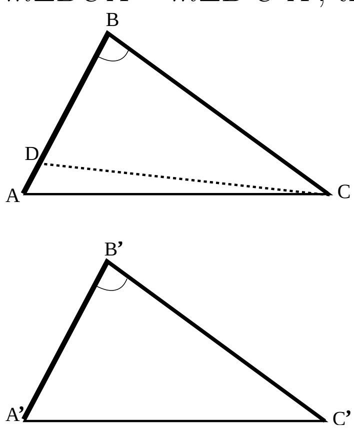
2.8. Monotonicity of lengths and angles. Here are two more axioms we shall need. Essentially, they say that for every real number, a segment can be scaled to that length, and that angles can be subdivided into angles of any measurement between 0 and π.

Axiom 6 (Ruler Axiom)**.** *If* A B *and* C *are distinct points on a line, in that order, then* $|AB| < |AC|$. *Further, for every positive real number* $r < |AC|$, *there is a point* B *between* A *and* C *so that* $|AB| = r$.

Axiom 7 (Protractor Axiom)**.** *If k is a line, and A is a point on k, then, for every number α with $0 < \alpha < \pi$, there is a line m through A so that the angles formed by k and m have measures α and $\pi - \alpha$. Further, if $0 < \beta < \alpha < \pi$, then there is a line n passing through the sector of angle α formed by k and m, so that n and k form an angle of measure β.*

Theorem 2.1 (SAS)**.** *If $\triangle ABC$ and $\triangle A'B'C'$ are such that $|AB| = |A'B'|$, $m\angle ABC = m\angle A'B'C'$, and $|BC| = |B'C'|$, then they are congruent.*

Proof. Suppose we are given two triangles $\triangle ABC$ and $\triangle A'B'C'$ as in the statement. If $m\angle BCA = m\angle B'C'A'$, then we would be done (by ASA).



So let us consider the case where they are different, and arrive at a contradiction. We may assume that $m\angle BCA > m\angle B'C'A'$ (if not, just exchange the names on the triangles).
Apply the second part of Axiom 7 to find a line passing through the point C and some point D lying between A and B, so that $m\angle BCD = m\angle B'C'A'$.
By ASA, $\triangle BCD \cong \triangle B'C'A'$. Therefore $|DB| = |A'B'|$. But we are given that $|A'B'| = |AB|$. Therefore, $|DB| = |AB|$. Since D lies on the line determined by A and B, and lies between them, this contradicts Axiom 6.

□

2.9. Isosceles triangles. A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem 2.2 (Base angles equal)**.** *If $\triangle ABC$ is isosceles, with base BC, then $m\angle B = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle B = m\angle C$, then it is isosceles, with base BC.

Exercise 2.5: Prove Theorem 2.2 by showing that $\triangle ABC$ is congruent to its reflection $\triangle ACB$. Note that there are two parts to the theorem, and so you need to give essentially two separate arguments.

2.10. Congruence via SSS.

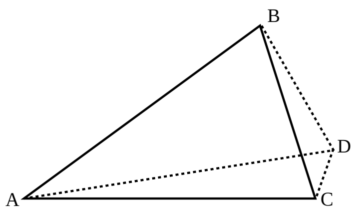
Theorem 2.3 (SSS)**.** *If $\triangle ABC$ and $\triangle A'B'C'$ are such that $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|BC| = |B'C'|$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. If the two triangles were not congruent, then one of the angles of $\triangle ABC$ would have measure different from the measure of the corresponding angle of $\triangle A'B'C'$. If necessary, relabel the triangles so that $\angle A$ and $\angle A'$ are two corresponding angles which differ, with $m\angle A' < m\angle A$.

We find a point D and construct the line AD so that $m\angle DAB = m\angle A'$, and $|AD| = |A'C'|$. (That this can be done follows from Axioms 6 and 7.) It is unclear where the point D lies: it could lie inside triangle ABC; it could lie on the line BC between B and C; or it could lie on the other side of the line BC. We need to take up these three cases separately.

Exercise 2.6: Suppose the point D lies on the line BC. Explain why this yields an immediate contradiction.

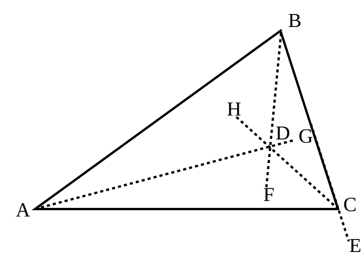
For both of the remaining cases, we draw the lines BD and CD. We observe that, by SAS, $\triangle ABD \cong \triangle A'B'C'$. It follows that $|BD| = |B'C'| = |BC|$ and that $|AD| = |A'C'| = |AC|$. Hence $\triangle BDC$ is isosceles, with base DC, and $\triangle ADC$ is isosceles with base CD. Since the base angles of an isosceles triangle have equal measure, $m\angle BDC = m\angle BCD$ and $m\angle ADC = m\angle ACD$.



First, we take up the case that D lies outside $\triangle ABC$; that is, D lies on the other side of the line BC from A.

Exercise 2.7: Finish this case of the proof, first by showing that $m\angle ADC < m\angle BDC$ and $m\angle BCD < m\angle ACD$. Then use the isosceles triangles to arrive at the contradiction that $m\angle ADC < m\angle ADC$.

We now consider the case where D lies inside $\triangle ABC$. Extend the line BC to some point E. Observe that $m\angle BCD + m\angle DCA + m\angle ACE = \pi$, from which it follows that $m\angle BCD + m\angle DCA < \pi$. Next, extend the line BD past D to some point F. Also extend the line AD past the point D to some point G, and extend the line CD past the point D to some point H.



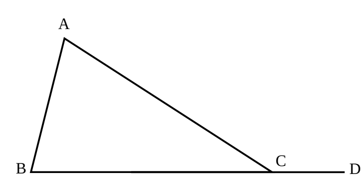
Exercise 2.8: Finish this case of the proof by explaining why $\pi < m\angle BDC + m\angle CDA$ and $m\angle BCD + m\angle DCA < \pi$, and then show that this leads to the contradiction $\pi < \pi$.

□

2.11. Inequalities for general triangles.

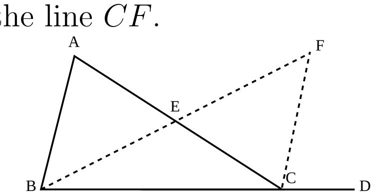
Theorem 2.4 (exterior angle inequality)**.** *Consider the triangle $\triangle ABC$. Let D be some point on the line BC, where C lies between B and D. Then*

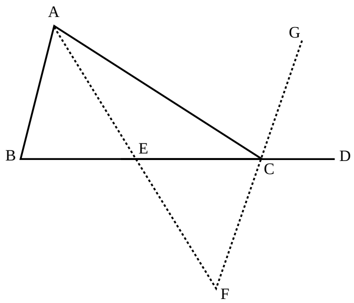
(i) $m\angle ACD > m\angle A$.
(ii) $m\angle ACD > m\angle B$.



Proof. We first prove part (i). Let E be the midpoint of the line segment AC; that is, E lies on the line AC, between A and C, and $|AE| = |EC|$. Draw the line BE and extend it past E to the point F, so that E is the midpoint of BF. Also draw the line CF.

Exercise 2.9: Finish the proof of part (i). Hint: First show that $\triangle AEB \cong \triangle CEF$ (Thm. 1.1 may be useful.) Use that to compare $m\angle A$ and $m\angle ECF$, and conclude that $m\angle ACD > m\angle ACF = m\angle A$.





For part (ii), we choose E to be the midpoint of the line BC, and extend AE beyond E to F, so that $|AE| = |EF|$. Also, extend the line Now extend the line FC beyond C to some point G.

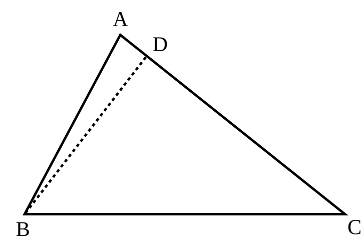
Exercise 2.10: Finish the proof of part (ii). First show that $\triangle AEB \cong \triangle FEC$, and then compare $m\angle FCE$, $m\angle DCG$, $m\angle DCA$, and $m\angle B$.

□

The next theorem says that in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle. This is generalizes the fact that the base angles of isosceles triangles are equal (Thm. 2.2).

Theorem 2.5. *In $\triangle ABC$, if $m\angle A > m\angle B$, then we must have $|BC| > |AC|$.*

Proof. Assume not. Then either $|BC| = |AC|$ or $|BC| < |AC|$.
Exercise 2.11: Show that if $|BC| = |AC|$, the assumption $m\angle A > m\angle B$ is contradicted.



Exercise 2.12: Now assume $|BC| < |AC|$, find the point D on AC so that $|BC| = |CD|$, and draw the line BD. Finish the proof in this case. Hint: Use Thm. 2.4 and the fact that $|BD| = |CD|$ to conclude that $m\angle CDB > m\angle A$. Now observe that $m\angle DBC < m\angle ABC$. Explain why this gives the contradiction $m\angle CBD < \angle CBD$.

□

The converse of the previous theorem is also true: opposite a long side, there must be a big angle.

Theorem 2.6. *In $\triangle ABC$, if $|BC| > |AC|$, then $m\angle A > m\angle B$.*

Proof. Assume not. If $m\angle A = m\angle B$, then $\triangle ABC$ is isosceles, with apex at C, so $|BC| = |AC|$, which contradicts our assumption.

If $m\angle A < m\angle B$, then, by the previous theorem, $|BC| < |AC|$, which again contradicts our assumption. □

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 2.7 (the triangle inequality)**.** *In $\triangle ABC$, we have*

$$|AB| + |BC| > |AC|$$

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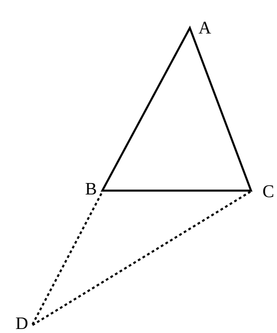


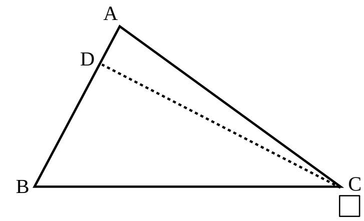
Exercise 2.13: Prove the triangle inequality: First extend AB to a point D so that $|BD| = |BC|$, then form the isosceles triangle $\triangle BDC$. Use this triangle and Thm 2.2 to show that $m\angle ADC < m\angle ACD$. Conclude that $|AD| > |AC|$ by using another theorem from this section. Then show that $|AB| + |BC| > |AC|$.

2.12. Congruence via AAS.

Theorem 2.8 (AAS). *Suppose we are given triangles ABC and $A'B'C'$, where $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, and $|BC| = |B'C'|$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Proof. We first observe that, by either SAS or ASA, if $|AB| = |A'B'|$, then $\triangle ABC \cong \triangle A'B'C'$. Hence we can assume that $|AB| \neq |A'B'|$, from which it follows that either $|AB| < |A'B'|$ or $|AB| > |A'B'|$. We can assume without loss of generality that $|AB| > |A'B'|$ (that is, if we had that $|AB| < |A'B'|$, then we would interchange the labelling of the two triangles).

Now find the point D between A and B, so that $|BD| = |A'B'|$. Observe that, by SAS, $\triangle DBC \cong \triangle A'B'C'$. Hence $m\angle BDC = m\angle A' = m\angle A$. This contradicts that fact that, since $\angle BDC$ is an exterior angle for $\triangle ADC$, we must have that $m\angle BDC > m\angle A$.



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This concludes the generalities concerning congruence of triangles. We now know the four congruence theorems, ASA, SAS, SSS and AAS. We also know that the other two possibilities, SSA and AAA, are not valid. It follows that, for example, if we are given the lengths of all three sides of a triangle, then the measures of all three angles are determined. However, we do not as yet have any means of computing the measures of these angles in terms of the lengths of the sides.

2.13. Perpendicularity and orthogonality.

Two lines intersecting at a point A are perpendicular or orthogonal if all four angles at A are equal. In this case, each of the angles has measure $\pi/2$. These angles are called right angles. It is standard in mathematics to use the words perpendicular and orthogonal interchangeably.

Basic Construction. Given a line k, and any point A, there is a line through A perpendicular to k.

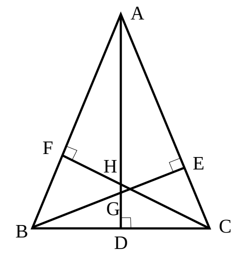
Exercise 2.14: Prove that the line through A perpendicular to k is unique. (Note that A may or may not lie on k.)

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC; the median from A bisects BC (that is, it crosses BC at a point D so that $|BD| = |DC|$); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

Theorem 2.9. *If A is the apex of the isosceles triangle ABC, and AD is the altitude, then AD is also the median, and is also the angle bisector, from A.*

Exercise 2.15: Prove this theorem. (Hint: Construct the altitude and apply AAS to the pair of resulting triangles.)

Theorem 2.10. *In an isosceles triangle, the three altitudes meet at a point.*



Proof. Let A be the apex of the isosceles $\triangle ABC$, and let AD be the altitude, which is also the median and the angle bisector. Similarly, let E be the endpoint on AC of the altitude from B, and let F be the endpoint on AB of the altitude from C. Let G be the point of intersection of AD with BE, and let H be the point of intersection of AD with CF. We need to prove that $G = H$.

By AAS, $\triangle FAC \cong \triangle EAB$. Hence $|AF| = |AE|$. Since AD is also the angle bisector, by ASA, $\triangle AFH \cong \triangle AEG$. Hence $|AH| = |AG|$, from which it follows that $G = H$. □

Exercise 2.16: Prove that the three angle bisectors in an isosceles triangle meet at a point.

Exercise 2.17: Prove that the three medians in an isosceles triangle meet at a point.

3. The parallel axiom

Axiom 8 (Parallel Axiom)**.** *Given a line k, and a point A not on k, there is exactly one line m passing through A and parallel to k.*

We remark that the point of the axiom is not the existence of the parallel, but the uniqueness. We will see below that existence actually follows from what we already know.

It is sometimes convenient to think of a line as being parallel to itself, so we make the following formal definition. Two lines are not parallel if they have exactly one point in common; otherwise they are parallel.

Theorem 3.1. *In the set of all lines in the plane, the relation of being parallel is an equivalence relation.*

Proof. First, since a line has infinitely many points in common with itself, it is parallel to itself; hence the relation is reflexive (this is the point of the strange definition).

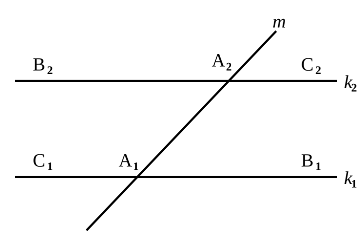
Second, the definition is obviously symmetric; it is defined in terms of the two lines; not one with relation to the other.

Third, suppose k is parallel to m, and m is parallel to n. There is obviously nothing further to prove unless the three lines are distinct. Assume that k and n are not parallel. Since two lines are either equal, parallel, or have exactly one point in common, we must have that k and n have a point in common. But this contradicts Axiom 8. □

3.1. Alternate interior angles.

We will meet the following situation some number of times. We are given two lines k_1 and k_2, and a third line m, where m crosses k_1 at A_1 and m crosses k_2 at A_2. Choose a point $B_1 \neq A_1$ on k_1, and choose a point $B_2 \neq A_2$ on k_2, where B_1 and B_2 lie on opposite sides of the line m. Then $\angle B_1A_1A_2$ and $\angle B_2A_2A_1$ are referred to as alternate interior angles.

In any given situation, there are two distinct pairs of alternate interior angles. That is, let C_1 be some point on k_1, where B_1 and C_1 lie on opposite sides of m, and let C_2 be some point on k_2, where C_2 and B_2 lie on opposite sides of m. Then one could also regard $\angle C_1A_1A_2$ and $\angle C_2A_2A_1$ as being alternate interior angles. However, observe that $m\angle B_1A_1A_2 + m\angle C_1A_1A_2 = \pi$ and $m\angle B_2A_2A_1 + m\angle C_2A_2A_1 = \pi$. It follows that one pair of alternate interior angles are equal if and only if the other pair of alternate interior angles are equal.



Proposition 3.2. *If the alternate interior angles are equal, then the lines k_1 and k_2 are parallel.*

Proof. Suppose not. Then the lines k_1 and k_2 meet at some point D. If necessary, we interchange the roles of the B_i and the C_i so that $\angle B_1A_1A_2$ is an exterior angle of $\triangle A_1A_2D$. Then D and B_2 lie on the same side of m, so $\angle DA_2A_1 = \angle B_2A_2A_1$. By the exterior angle inequality,

$$m\angle B_1A_1A_2 > m\angle A_1A_2D = m\angle B_2A_2A_1 = m\angle B_1A_1A_2,$$

so we have reached a contradiction. □

3.2. Existence of parallel lines. Let k_1 be a line, and let A_2 be a point not on k_1. Pick some point A_1 on k_1 and draw the line m through A_1 and A_2. By Axiom 7, we can find a line k_2 through A_2 so that the alternate interior angles are equal. Hence we can find a line through A_2 parallel to k_1.

Theorem 3.3 (alternate interior angles equal)**.** *Two lines k_1 and k_2 are parallel if and only if the alternate interior angles are equal.*

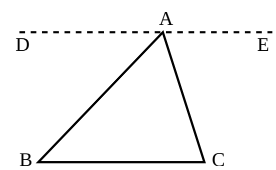
Proof. To prove the forward direction, construct the line k_3 through A_2, where there is a point B_3 on k_3, with B_3 and B_2 on the same side of m, so that $m\angle B_3A_2A_1 = m\angle B_1A_1A_2$. Then, by Prop. 3.2, k_3 is a line through A_2 parallel to k_1. Axiom 8 implies $k_3 = k_1$. Hence $m\angle B_3A_2A_1 = m\angle B_2A_2A_1$, and the desired conclusion follows.

The other direction is just Prop. 3.2, restated as part of this theorem for convenience. □

3.3. The sum of the angles of a triangle.

Theorem 3.4. *The sum of the measures of the angles of a triangle is equal to π.*

Proof. Consider $\triangle ABC$, and let m be the line passing through A and parallel to BC. Let D and E be two points on m, on opposite sides of A, where D and C lie on opposite sides of the line AB. Then B and E lie on opposite sides of AC.



Exercise 3.1: Use alternate interior angles to complete the proof of this theorem.

□

A quadrilateral is a region bounded by four line segments, so it has four verticies on its boundary.

Corollary 3.5. *The sum of the measures of the angles of a quadrilateral is 2π.*

Proof. Cut the triangle into two triangles, and do the obvious computation. □

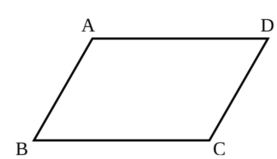
A rectangle is a quadrilateral in which all four angles are right angles.

Theorem 3.6. *If $ABCD$ is a rectangle, then AB is parallel to CD, and $|AB| = |CD|$. Similarly, BC is parallel to AD and $|BC| = |AD|$.*

Exercise 3.2: Prove this theorem.

i. Prove that opposite pairs of sides are parallel.
ii. Now cut the rectangle into two triangles; prove that these two triangles are congruent. Conclude that opposite sides of the rectangle have equal length.

Somewhat more generally, a parallelogram is a quadrilateral $ABCD$ in which opposite sides are parallel; that is, AB is parallel to CD, and AD is parallel to BC.



A rectangle with all four sides of equal length is a square; a parallelogram with all four sides of equal length is a rhombus.

Theorem 3.7. *Let $ABCD$ be a parallelogram. Then $m\angle A = m\angle C$; $m\angle B = m\angle D$; $|AB| = |CD|$; and $|BC| = |AD|$.*

Exercise 3.3: Prove this theorem. (Hint: Draw a diagonal.)

Theorem 3.8. *If $ABCD$ is a quadrilateral in which $|AB| = |CD|$ and $|AD| = |BC|$, then $ABCD$ is a parallelogram.*

Exercise 3.4: Prove this theorem.

Theorem 3.9. *Let $ABCD$ be a parallelogram with diagonals of equal length (that is, $|AC| = |BD|$). Then $ABCD$ is a rectangle.*

Exercise 3.5: Prove this theorem.

4. Lengths, areas and proportions

We now turn to a brief discussion of area. Rather than carefully developing this theory, we shall begin with some "obvious" facts, such as

(i) The area of a rectangle which has side lengths a and b is ab.
(ii) Congruent figures have equal area
(iii) The area of a region obtained as a union of two non-overlapping regions is the sum of the areas of these regions.

4.1. Right triangles. A *right triangle* is one which contains a right angle. The two sides emanating from the right angle are called the *legs* of the triangle; the third side, opposite the right angle is the *hypoteneuse*.

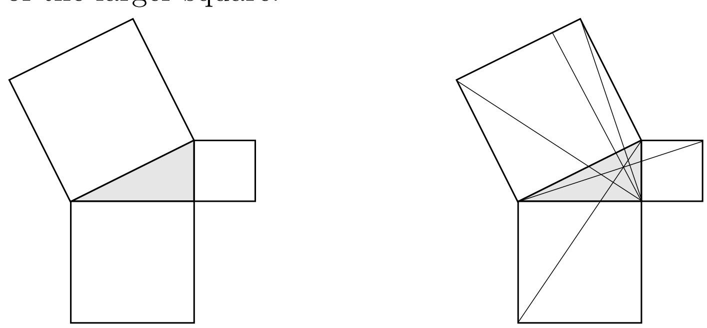
Proposition 4.1. *If the legs of a right triangle have lengths a and b, then the area of the triangle is $\frac{1}{2}ab$.*

Exercise 4.1: Prove this proposition by constructing an appropriate rectangle.

Proposition 4.2. *In $\triangle ABC$, if D is the endpoint of the altitude from A, then the area of $\triangle ABC$ is equal to $\frac{1}{2}|AD||BC|$.*

Exercise 4.2: Prove this proposition.

The Pythagorean Theorem, relating the lengths of the legs of a right triangle to the length of the hypotenuse, is very well-known; you've probably seen it repeatedly since your elementary school days, and know it quite well. The figure on the left below is sometimes described as a proof of the Pythagorean Theorem, but in fact it is much closer to a geometric statement of the theorem. To turn it into a proof requires constructing a number of additional lines, and a fairly complex argument. On left is the figure for Euclid's proof, sometimes called "the bride's chair". We won't give the argument here, but you might try to figure it out on your own. The idea is to show that there are two pairs of congruent triangles, and the area of each triangle in a pair is half of one of the smaller squares. The sum of the areas in the pairs gives the area of the larger square.

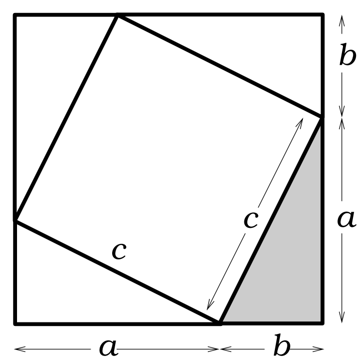


Theorem 4.3 (The Pythagorean Theorem)**.** *Let the lengths of the legs of a right triangle be a and b, and let the length of the hypoteneuse be c. Then $a^2 + b^2 = c^2$.*

There are many proofs of this theorem; we know of at least 40. Below we give one of the simpler ones. It is necessary to use the Parallel Axiom (Axiom 8) axiom, either implicitly or explicitly, in order to prove the Pythagorean Theorem; the theorem is false in both spherical and hyperbolic geometry, which have a different version of Axiom 8. Surprisingly, the Pythagorean Theorem is equivalent to this axiom; that is, the Parallel Axiom can be proven if we assume the Pythagorean Theorem first.

Proof. The proof is essentially the figure shown. We start with a right triangle with legs of length a and b and hypotenuse of length c. Now we construct three more copies of this triangle, arranging them to construct a square of side length $a + b$ as in the figure. This is indeed a square because all four sides are of length $a + b$, and each angle of the outer quadrilateral is a right angle.

In addition, there is an inner quadrilateral formed whose sides are the hypotenuse of each of the triangles. This quadrilateral is certainly a rhombus, because each side is of length c. But it is in fact a square, because each of its angles are right: at each vertex there are three angles which sum to an angle of measure π. Two of these are the acute angles in a right triangle, and so the third must be of measure $\pi/2$.



The area of the outer square is $(a + b)^2$, the area of each of the four triangles is $\frac{1}{2}ab$, and the area of the inner square is c^2. Thus, we have

$$(a + b)^2 = 4(\frac{1}{2}ab) + c^2$$

From this, we readily see that $a^2 + b^2 = c^2$. □

Exercise 4.3: Locate a different proof of the Pythagorean Theorem, and explain it in your own words.

4.2. Similar triangles. We say that two triangles ABC and $A'B'C'$ are similar if $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ and $m\angle C = m\angle C'$.

Since the sum of the measures of a triangle is constant, if we are given two triangles, $\triangle ABC$ and $\triangle A'B'C'$, and we know that $m\angle A = m\angle A'$, and $m\angle B = m\angle B'$, then the triangles are similar.

If the triangles are similar, we write $\triangle ABC \sim \triangle A'B'C'$.

Proposition 4.4. *Let D be some point on the side AB of $\triangle ABC$, and let k be the line through D parallel to BC. Let E be the point where k crosses AC. Then $\triangle ABC \sim \triangle ADE$.*

Exercise 4.4: Prove this proposition.

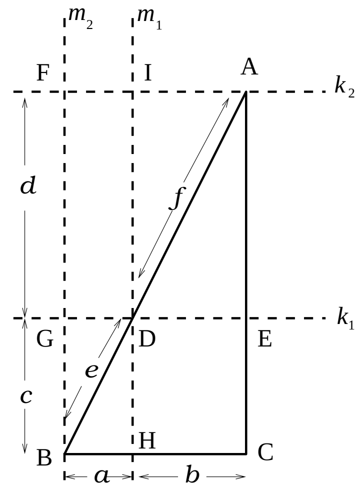
We want to prove the very useful fact that the side lengths of similar triangles are proportional. However, this is easier to do if we first prove it for right triangles, and then apply this result to the general case.

Lemma 4.5 (Ratios for right triangles)**.** *Let $\triangle ABC \sim \triangle A'B'C'$, where C and C' are right angles. Then*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

Proof. Without loss of generality, we can assume that $|AB| > |A'B'|$, and so we start with $\triangle ABC$, and construct another triangle which is congruent to $\triangle A'B'C'$ inside it.

Let k_1 be a line parallel to BC that meets AB at the point D where $|AD| = |A'B'|$. Denote the point where k_1 meets AC by E. Let k_2 be the line through A parallel to BC, and let m_2 be the line perpendicular to AC at B. Then, since AC and m_2 are both perpendicular to BC, they are parallel. Let F be the point of intersection of m_2 with k_2. Then $AFBC$ is a rectangle. Also, let G be the point of intersection of m_2 and k_1. Finally, let m_1 be the line perpendicular to k_1 and k_2 through D; denote the point of intersection of m_1 with BC by H and denote the point of intersection of m_1 with k_2 by I.



Since the lines BC, k_1 and k_2 are all parallel, and the lines AC, m_1 and m_2 are all parallel, $|BH| = |GD| = |FI|$, $|HC| = |DE| = |IA|$, $|BG| = |HD| = |CE|$, and $|GF| = |DI| = |EA|$. We give names to these quantities; we set $a = |BH| = |GD| = |FI|$, $b = |HC| = |DE| = |IA|$, $c = |BG| = |HD| = |CE|$ and $d = |GF| = |IE| = |EA|$. We also set $e = |BD|$ and $f = |DA|$.

The line AB divides the rectangle $AFBC$ into two congruent triangles. Since the areas of $\triangle BGD$ and $\triangle BHD$ are equal, and the areas of $\triangle IDA$ and $\triangle EAD$ are equal, we obtain that the areas of the two smaller rectangles are equal; that is, $ad = bc$.

Since we drew the line EG parallel to BC, we know that $\triangle AED \sim \triangle ACB$. So, for this case, our theorem, which we want to prove, says that

$$\frac{d}{c+d} = \frac{b}{a+b} = \frac{f}{e+f}.$$

To see that the first of these inequalities is true, notice that it holds when $d(a+b) = b(c+d)$ (by cross multiplying). But this holds since we have already established that $ad = bc$.

We check that the second equality is true by using the first equality together with the Pythagorean theorem. That is, we write $f^2 = b^2 + d^2$ and $(e+f)^2 = (a+b)^2 + (c+d)^2$, and then, as above, cross-multiply and use the two facts that we have already proven; namely that $ad = bc$, and that

$$\frac{d^2}{(c+d)^2} = \frac{b^2}{(a+b)^2}.$$

Now since $\triangle ABC \sim \triangle A'B'C'$, we have $m\angle A = m\angle A'$ and $m\angle B = m\angle B'$. Since k_1 is parallel to BC, $m\angle B = m\angle EDA$ (using alternate interior angles and vertical angles). Since $|AD| = |A'B'|$ by construction, $\triangle A'B'C' \cong \triangle ADE$ via ASA. Using the above, we conclude that

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}$$

as desired. □

Now that we have shown the desired property for right triangles, we can use that result to show it for arbitrary similar triangles.

Theorem 4.6 (Ratios for triangles)**.** *Let $\triangle ABC \sim \triangle A'B'C'$. Then*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

Proof. As above, we can assume without loss of generality that $|AB| > |A'B'|$. Find the point D on AB so that $|AD| = |A'B'|$, and draw the line DE parallel to BC. As above, using alternate interior angles, we see that $m\angle ADE = m\angle ABC$ and $m\angle AED = m\angle ACB$. Hence $\triangle ADE \cong \triangle A'B'C'$, and so it suffices to show that

$$\frac{|AB|}{|AD|} = \frac{|BC|}{|DE|} = \frac{|CA|}{|EA|}.$$

Let $k = AF$ be the altitude from A, and let G be the point of intersection of k with the line DE. Then ADF and ADG are similar right triangles, and ACF and AEG are similar right triangles. Hence the proportion theorem for right triangles yields

$$\frac{|AD|}{|AB|} = \frac{|AG|}{|AF|} = \frac{|AE|}{|AC|}.$$

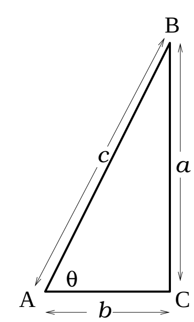
That $\frac{|DE|}{|BC|} = \frac{|AG|}{|AF|}$ follows easily from the Lemma above. □

Exercise 4.5: Prove that the lines joining the midpoints of the sides of a triangle divide the triangle into four congruent triangles.

4.3. Basic trigonometric functions. Let $\triangle ABC$ be such that $\angle C$ is a right angle. Let $a = |BC|$, $b = |AC|$ and $c = |AB|$. Also, let $\theta = m\angle A$. Then, by the proportion theorem for right triangles, the ratios a/b, a/c, and b/c all depend only on the measure of the angle θ.

In what follows, we will deliberately confuse an angle with its measure; i.e., we will not distinguish between $\angle A$ and $m\angle A$. It should always be clear from the context which is meant.

For $0 < \theta < \pi/2$, we define the trigonometric functions as ratios of lengths in right triangles:



$\sin\theta = a/c$
$\cos\theta = b/c$
$\tan\theta = a/b = \sin\theta/\cos\theta$
$\cot\theta = b/a = 1/\tan\theta = \cos\theta/\sin\theta$
$\sec\theta = c/b = 1/\cos\theta$
$\csc\theta = c/a = 1/\sin\theta$

We note that it is almost immediate from the definition that

$$\sin(\pi/2 - \theta) = \cos\theta \qquad \text{and} \qquad \cos(\pi/2 - \theta) = \sin\theta.$$

In order to extend the definitions to all values of θ, first we set

$$\sin 0 = 0 \qquad \text{and} \qquad \sin\frac{\pi}{2} = 1$$

which is reasonable since the sine is small for small angles, and close to 1 for angles close to right angles. Now to define obtuse angles,

$$\sin(\pi - \alpha) = \sin(\alpha) \qquad \text{for} \qquad 0 \leq \alpha \leq \frac{\pi}{2}$$

and to get negative values, we let

$$\sin(-\theta) = -\sin(\theta).$$

Finally, since an angle of 2π represents a full turn, we say that $\sin(2\pi + \alpha) = \sin\alpha$. Putting these together defines the sine for all values.

We can then use the fact that $\sin(\pi/2 - \theta) = \cos\theta$ and $\tan\theta = \sin\theta/\cos\theta$ to extend the definitions of the other functions to all real numbers (except those where we would need to divide by 0, such as for $\tan\frac{\pi}{2}$.

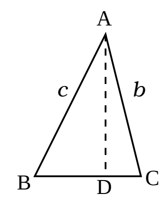
Exercise 4.6: Prove that, for any number θ,

$$\sin^2\theta + \cos^2\theta = 1.$$

Theorem 4.7 (Law of Sines)**.** *Let ABC be a triangle, with sides of length $a = |BC|$, $b = |AC|$ and $c = |AB|$. Then*

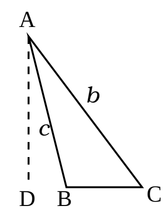
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Proof. We first assume that C is a right angle, and, that $c = 1$. Then $\sin A = a$ and $\sin B = b$, so the result holds in this case. Using the theorem of proportions for right triangles, our result holds for all right triangles.



Now let ABC be an arbitrary triangle, where B and C are acute angles (i.e., their measures are less than $\pi/2$), and let AD be the altitude from A. Then $\sin B = \frac{|AD|}{c}$, and $\sin C = \frac{|AD|}{b}$. Hence

$$\frac{\sin B}{b} = \frac{|AD|}{bc} = \frac{\sin C}{c}.$$



We next take up the case that angle B is an obtuse angle. We again drop the altitude AD from A to the line BC, but now this altitude lies outside $\triangle ABC$. Note that B lies between D and C. As above, using our known facts about right triangles, we obtain, $|AD| = |AC|\sin C = |AB|\sin B$, from which the desired result follows.

If angle C is obtuse, repeat the above argument, exchanging B and C.

The other equality follows by looking at the altitude from B and/or C. □

Theorem 4.8 (Law of Cosines)**.** *Let ABC be a triangle, where none of the angles are right angles, with sides of length $a = |BC|$, $b = |AC|$ and $c = |AB|$. Then*

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Proof. As above, we need to worry about the possibility that $\angle C$ is obtuse. We first take up the case that it is acute. It could be that either $\angle A$ or $\angle B$ is obtuse, we note that our hypotheses, and the formula we wish to prove, remain unchanged if we interchange A and B; we assume that $\angle B$ is not obtuse. It then follows that if AD is the altitute from A, then D lies between A and B. Let $d = |AD|$. Let $a_1 = |BD|$, and let $a_2 = |DC|$. Then, by the Pythagorean theorem, $c^2 = a_1^2 + d^2$. Again by the Pythagorean theorem, $d = b^2 - a_2^2$. Substituting, we obtain

$$c^2 = a_1^2 + d^2 = a_1^2 + b^2 - a_2^2 = (a_1 + a_2)^2 + b^2 - 2a_1a_2 - 2a_2^2 = a^2 + b^2 - 2aa_2 = a^2 + b^2 - 2ab\cos C,$$

where, in the last equality, we have used the fact that $\cos C = a_2/b$.

We next take up the case that $\angle C$ is obtuse. In this case, the endpoint of the altitude AD is such that C lies between B and D. Similar to the above, we set $a_1 = |CD|$. Then the

Pythorean theorem yields the following.

$$(a+a_1)^2 + d^2 = c^2, \quad d^2 + a_1^2 = b^2.$$

Substituting, we obtain,

$$c^2 = (a+a_1)^2 + b^2 - a_1^2 = a^2 + b^2 + 2aa_1.$$

We note that $\cos(\pi - C) = a_1/b$. Hence, upon setting $\cos C = -\cos(\pi - C)$, we obtain,

$$c^2 = a^2 + b^2 + -2ab\cos(\pi - C) = a^2 + b^2 - 2ab\cos C.$$

□

4.4. Other important trigonometric formulae. We will not prove the addition formulae for sines and cosines; however, since they are important, we will state them.

Theorem 4.9.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Theorem 4.10.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

One can easily prove these theorems by constructing $\triangle ABC$, where $m\angle A = \alpha + \beta$, and the altitude from A divides this triangle into two triangles, $\triangle ABD$ and $\triangle ADC$, in such a way that $m\angle BAD = \alpha$ and $m\angle DAC = \beta$. Then, using the definitions of the sine and cosine for right triangles, the addition formula for the sine follows from the law of sines, and the addition formula for the cosine follows from the Pythagorean theorem and the law of cosines.

We can use the laws of sines and cosines to obtain information about triangles.

Exercise 4.7: This exercise tells you how to compute the other three pieces of information about a triangle if you are given SAS.

(a) Assume we are given the lengths a and b of triangle ABC, and we are given $\angle C$ (that is, we are given $\sin C$ and/or $\cos C$). Explain in words how we would find the length of the other side, and the other two angles. (Hint: Use the above theorems.)
(b) Find explicit formulas for these three pieces of information in explicit terms; that is, find a formula for c, and find formulas for the other two angles, in terms of a, b and either $\sin C$ or $\cos C$.

Exercise 4.8: Repeat the above for the congruence relation ASA. That is, if you are given the sines and or cosines of two of the angles, and given the length of the included side, find the sine or cosine of the other angle and find the lengths of the other two sides.

Exercise 4.9: Repeat the above exercise for AAS.

Exercise 4.10: Repeat the above exercise for SSS.

5. Circles and lines

5.1. Circles. A circle Σ is the set of points at fixed distance $r > 0$ from a given point, its center. The distance r is called the radius of the circle Σ.

The circle Σ divides the plane into two regions: the inside, which is the set of points at distance less than r from the center O, and the outside, which consists of all points having distance from O greater than r. Note that every line segment from O to a point on Σ has the same length r.

A line segment from O to a point on Σ is also called a radius; this should cause no confusion.

A line segment connecting two points of Σ is called a chord, if the chord passes through the center, then it is called a diameter.

As above, we also use the word diameter to denote the length of a diameter of Σ, that is, the number that is twice the radius.

Proposition 5.1. *A line k intersects a circle Σ in at most two points.*

Proof. Suppose we had three points, A, B and C, of intersection of k with Σ.

We first take up the case that k is a diameter. In this case, we would have at least two of the three points on the same side of O on k; hence we can suppose that A and B both lie on the same side of O. However, by the ruler axiom (Axiom 6), we must have $|OA| \neq |OB|$ since $A \neq B$. This contradicts our assumption that A and B both lie on Σ.

We next take up the case that k is not a diameter. We can assume that B lies between A and C on k. Draw the line segments, OA, OB, OC. Then OAB, OAC and OBC are triangles. In fact, since $|OA| = |OB| = |OC|$, they are isosceles triangles. Let α be the measure of the base angles of triangle OAB. Then it is also the measure of the base angle of $\triangle OAC$, and so it is also the measure of the base angle of $\triangle OBC$. Since the two base angles at B add up to π, we obtain that each of the three triangles have two right angles, which is impossible. □

Proposition 5.2. *Let AB be a chord of a circle Σ with center O. Then the perpendicular bisector of AB passes through O.*

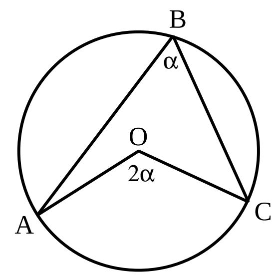
Exercise 5.1: Prove the preceding proposition.

5.2. Central angles. While we have spent a fair amount of time determining when two angles have the same measure, we have not discussed explicitly calculating the measure of an angle, except in the case of an angle of measure π and a right angle (measure $\pi/2$). We shall do so now.

First, we assume the well-known property that the circumference (that is, the arc length) of a circle of radius r is $2\pi r$.

Now let A and B be two points on a circle of radius 1 and center O. The radii AO and BO make two angles AOB (the "inner" and the "outer" angles); call them α and β. An angle such as α whose vertex is at the center of the circle is called a central angle. Notice that $\alpha + \beta = 2\pi$, no matter where A and B lie on Σ. If A and B are the endpoints of a diameter, they divide the circle into two arcs, each of length π; note also that the measure of the angles α and β are also π. In other cases, the length of the arc subtended by the angle α will be whatever fraction of 2π that α is of the entire circle. For example, if α is a right angle, it will take up 1/4 of the circle, and the corresponding arc length will be $\pi/2$. We define the measure of the angle to be the corresponding arc length when that angle is the central angle of a circle of radius 1.

Theorem 5.3 (measure of inscribed angle is half the central angle)**.** *Let A, B and C be points on the circle Σ of radius r. Draw the chords AB and BC, and draw the radii, OA and OC. Let α be the measure of the inscribed angle ABC. Then the measure of the central angle AOC is 2α. (Here we mean the angle AOC which subtends the arc not containing B.)*

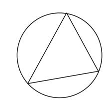


Proof. Draw the line OB. This divides quadrilateral $ABCO$ into two isosceles triangles. Let β be the measure of the base angles of $\triangle OAB$, and let γ be the measure of the base angles of $\triangle OBC$ Then the measure of the requisite central angle is given by

$$m\angle AOC = 2\pi - (\pi - 2\beta) - (\pi - 2\gamma) = 2(\beta + \gamma) = 2m\angle ABC = 2\alpha$$

□

5.3. Circumscribed circles. The circle Σ is circumscribed about $\triangle ABC$ if all three vertices of the triangle lie on the circle. In this case, we also say that the triangle is inscribed in the circle.



Note that another way to describe a circle circumscribed about a triangle is to say that it is the smallest circle for which every point inside the triangle is also inside the circle. In this view, the problem of circumscribing a circle becomes a minimization problem. A given triangle lies inside many circles, but the circumscribed circle is, in some sense, the smallest circle which lies outside the given triangle.

It is not immediately obvious that one can always solve this minimization problem, nor that the solution is unique.

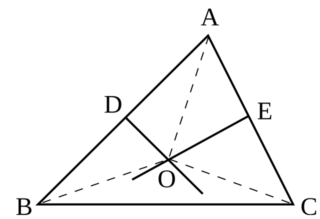
Proposition 5.4 (Uniqueness of Circumscribed Circles)**.** *There is at most one circle circumscribed about any triangle.*

Proof. Suppose there are two circles Σ and Σ' which are circumscribed about $\triangle ABC$. Since points A, B, and C lie on both circles, AB and BC are chords. By Prop. 5.2, the perpendicular bisectors of AB and BC both pass through the centers of Σ and Σ'. Since these two distinct lines can intersect in at most one point, Σ and Σ' share the same center O. Since AO is a radius for both circles, they have the same center and radius, and hence are the same circle. □

Theorem 5.5 (Existence of Circumscribed Circles)**.** *Given $\triangle ABC$, there is always exactly one circle Σ circumscribed about it.*

Proof. We need to show that the perpendicular bisectors of the sides of $\triangle ABC$ meet at a point, and that this point is equidistant from all three vertices. Then the requisite circle will have this point as its center O, and the radius will be the length of AO. Uniqueness was shown in Prop. 5.4.

Let D and E be the midpoints of sides AB and BC respectively. Draw the perpendicular bisectors of AB and BC, and let O be the point where these two lines meet (note that O need not be inside the triangle). Draw the lines AO, BO and CO.
We cannot have both that $O = D$ and $O = E$ (since $D \neq E$), hence we can assume without loss of generality that $O \neq D$. Then we have $|AD| = |DB|$, angles $\angle ADO$ and $\angle BDO$ are both right angles, and of course, $|DO| = |DO|$. Hence, $\triangle ADO \cong \triangle BDO$ by SAS. In particular, $|AO| = |BO|$. If $O = E$, then we have shown that $|AO| = |BO| = |CO|$, from which it follows that there is a circumscribed circle with center O and radius $|AO|$.



If $O \neq E$, then we repeat the above argument to show that $\triangle BOE \cong \triangle COE$, from which, as above, it follows that $|OB| = |OC|$. Again, this shows that there is a circumscribed circle. □

Corollary 5.6. *In any triangle, the three perpendicular bisectors of the sides meet at a point.*

Exercise 5.2: Explain why Theorem 5.5 implies this corollary.

Corollary 5.7 (Three Points Determine a Circle)**.** *Given any three non-colinear points, there is exactly one circle which passes through all three of them.*

Exercise 5.3: Explain why this corollary follows from Theorem 5.5.

5.4. Tangent lines and inscribed circles. A line that meets a circle in exactly one point is a tangent line to the circle at the point of intersection. Our first problem is to show that there is one and only one tangent line at each point of a circle.

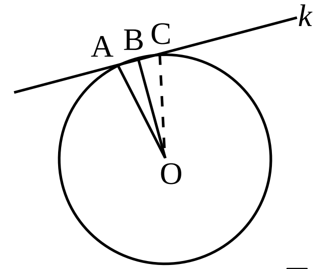
Proposition 5.8. *Let A be a point on the circle Σ, and let k be the line through A perpendicular to the radius at A. Then k is tangent to Σ.*

Proof. There are only three possibilities for k: it either is disjoint from Σ, which cannot be, as A is a common point; or it is tangent to Σ at A; or it meets Σ at another point B. If k meets Σ at B then OAB is a triangle, where $\angle A$ is a right angle. Since OA and OB are both radii, $|OA| = |OB|$. Hence $\triangle OAB$ is isosceles. Hence $m\angle A = m\angle B$. We have constructed a triangle with two right angles, which cannot be; i.e., we have reached a contradiction. □

Proposition 5.9. *If k is a line tangent to the circle Σ at the point A, then k is perpendicular to the radius ending at A.*

Proof. We will prove the contrapositive: if k is a line passing through A, where k is not perpendicular to the radius, then k is not tangent to Σ.

Draw the line segment m from O to k, where m is perpendicular to k. Let B be the point of intersection of k and m. On k, mark off the distance $|AB|$ from B to some point C, on the other side of B from A. Since OB is perpendicular to k, $m\angle OBA = m\angle OBC$. By SAS, $\triangle OBA \cong \triangle OBC$, and so $|OC| = |OA|$. Thus both A and C lie on Σ, and k intersects Σ in two points. Thus, k is not tangent to Σ.

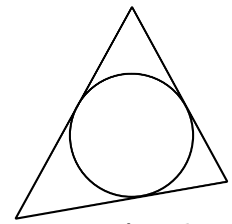


□

Corollary 5.10. *Let A be a point on the circle Σ. Then there is exactly one line through A tangent to Σ.*

Exercise 5.4: Prove this Corollary.

A circle Σ is inscribed in $\triangle ABC$ if all three sides of the triangle are tangent to Σ. One can view the inscribed circle as being the largest circle whose interior lies entirely inside the triangle. (Note that it is not quite correct to say that the circle lies entirely inside the triangle, because the triangle and the circle share three points.)



We start the search for the inscribed circle with the question of what it means for the circle to have two tangents which are not parallel.

Proposition 5.11. *Let A be a point outside the circle Σ, and let k_1 and k_2 be tangents to Σ emanating from A. Then the line segment OA bisects the angle between k_1 and k_2.*

Proof. Let B_i be the point where k_i is tangent to Σ, for $i = 1, 2$. Draw the lines OB_1 and OB_2. Observe that $|OB_1| = |OB_2|$, and that, since radii are perpendicular to tangents, $\angle OB_1A = \angle OB_2A$, and these are both right angles.

By SSA, $\triangle OB_1A \cong \triangle OB_2A$. Hence $m\angle OAB_1 = m\angle OAB_2$. □

From the above, we see that if there is an inscribed circle for $\triangle ABC$, then its center lies at the point of intersection of the three angle bisectors, and its radius is the distance from this point to the three sides. Hence we have proven the following.

Corollary 5.12 (Inscribed circles are unique)**.** *Every triangle has at most one inscribed circle.*

Theorem 5.13. *Every triangle has an inscribed circle.*

Proof. Let G be the point of intersection of the angle bisectors from A and B in $\triangle ABC$. Let D be the point where the orthogonal from G meets AB; let E be the point where the orthogonal from G meets BC; and let F be the point where the orthogonal from G meets AC.

Observe that, by AAS, $\triangle ADG \cong \triangle AFG$. Similarly, $\triangle BDG \cong \triangle BEG$ and $\triangle CEG \cong \triangle CFG$.

We have shown that the perpendiculars from G to the three sides all have equal length; call this length r. then the circle centered at G of radius r is tangent to the three sides of $\triangle ABC$ exactly at the points D, E and F. □

This theorem gives another proof of the result of exercise 2.16.

Corollary 5.14. *The three angle bisectors of a triangle meet at a point; this point is the center of the inscribed circle.*

Exercise 5.5: Give a proof of this corollary using the above theorem.

Exercise 5.6: Let $\triangle ABC$ and $\triangle A'B'C'$ be such that $|AB| = |A'B'|$, $|BC| = |B'C'|$, and $m\angle C = m\angle C' = \pi/2$. Prove that $\triangle ABC \cong \triangle A'B'C'$.

Exercise 5.7: Let A and B be points on the circle Σ. Let k be the line tangent to Σ at A and let m be the line tangent to Σ at B. Prove that if k and m are parallel, then the line segment AB is a diameter of Σ.

6. Some Amusements

6.1. Amusement 1: Fill in the steps in the construction below, and observe the result.

(1) Start with arbitrary triangle $\triangle T_1 = \triangle ABC$.
(2) Construct the lines k, m and n, parallel to AB, BC and CA, respectively.
(3) These three lines form a new triangle, $\triangle T_0 = \triangle A'B'C'$; label these so that $B'C'$ is parallel to BC, $A'C'$ is parallel to AC and $A'B'$ is parallel to AB.
(4) Observe that the sides of $\triangle T_1$ divide $\triangle T_0$ into four triangles.
(5) Since the sides are parallel, these four triangles are all similar to the big triangle; in particular, $\triangle A'B'C' \sim \triangle ABC$.
(6) Since they have some sides in common, the four smaller triangles are all congruent.
(7) It follows that A is the midpoint of $B'C'$; B is the midpoint of $A'C'$; and C is the midpoint of $A'B'$.
(8) Construct the perpendicular bisectors of $A'B'$, $B'C'$ and $A'C'$; we know these all meet at a point ; call it O. (O is the center of the circumscribed circle for $\triangle A'B'C'$; this center is called the *orthocenter*).
(9) The lines OA, OB and OC, when extended, are altitudes of $\triangle ABC$.
(10) We have shown that the three altitudes of an arbitrary triangle meet at a point.

6.2. Amusement 2:

(1) Let $\triangle ABC$ be an arbitrary triangle.
(2) Let AD and BE be medians. Let G be the point of intersection of these two lines.
(3) Draw the line DE.
(4) Observe that DE is parallel to AB. (This was part of a homework assignment.)
(5) Then $\triangle GAB \sim \triangle GDE$.
(6) We know that $|DE| = 1/2|AB|$. Hence, $|GD| = 1/2|AG|$ and $|GE| = 1/2|GB|$.
(7) Repeat the above argument, using the medians from A and C.
(8) Conclude that the three medians of an arbitrary triangle meet at a point. (This point is called the *centroid* of the triangle; it is at the center of gravity.)
(9) We also have shown that the centroid divides each median into two segments; the segment between the centroid and the vertex is twice as long as the segment between the centroid and the opposite side.

6.3. Circles and circles. Two circles Σ and Σ' either are disjoint, or they meet at a point, in which case they are said to be *tangent*, or they meet at two points, in which case, they *intersect*.

It is essentially immediate that two circles with the same center but different radii are disjoint.

Since a line is the shortest distance between two points, if we have two circles where the distance between the centers is greater than the sum of their radii, then the circles are necessarily disjoint.

If we have two circles with the property that the distance between their centers is exactly equal to the sum of their radii, then the line between their centers contains a point on both circles.

Proposition 6.1. *If two circles have three points in common, then they are identical.*

Proof. Label the three points as A, B and C, and draw the lines AB, BC and CA.

The three points cannot be collinear, for a line intersects a circle in at most two points. Since the three points are not collinear they form a triangle. Then both circles are circumscribed about $\triangle ABC$. Since the circumscribed circle about a triangle is unique, the two circles are the same. □

Proposition 6.2. *Suppose the circles Σ and Σ' intersect at the points A and B. Let O be the center of Σ and let O' be the center of Σ'. Then the line OO' is the perpendicular bisector of the line segment AB.*

Proof. Since AB is a chord of Σ (Σ'), the perpendicular bisector of the chord AB passes through the center O (O'). Hence the perpendicular bisector of the line segment AB is the line determined by O and O'. □

Proposition 6.3. *Suppose the circles Σ and Σ' are tangent at A. Then the line connecting the centers of these circles, passes through A.*

Proof. Let O be the center of Σ and let O' be the center of Σ'. Suppose the line OO' does not pass through A. Construct the perpendicular from A to the line OO', and let B be the point where this perpendicular bisector meets OO'. Now construct the point C on the line AB so that B lies between A and C and so that $|AB| = |BC|$. Then, by *sas*, $\triangle OAB \cong \triangle OCB$. Hence $|OC| = |OA|$, from which it follows that C lies on Σ. Using the same argument, $\triangle O'AB \cong \triangle O'CB$, from which it follows that C lies on Σ'. We have constructed a second point of intersection of Σ and Σ'; since we assumed these circles had only one point in common, we have reached a contradiction. □

Corollary 6.4. *If the circles Σ and Σ' are tangent at A, and k is the line tangent to Σ at A, then k is tangent to Σ'.*

Proof. Since Σ and Σ' are tangent at A the line OO' connecting their centers passes through A. Hence the radius of Σ at A lies on the line OO', and so OO' is orthogonal to k. The same argument shows that the radius of Σ' lies on the line OO'. Since the tangent to Σ' at A is the line orthogonal to the radius at A, it is k. □

We now return to the case of intersecting circles. Suppose the circles Σ and Σ' intersect at the points A and B. Then the line joining the centers O and O' is the perpendicular bisector of the line segment AB. We draw the radii, OA, OB, $O'A$ and $O'B$. We define the *angle of intersection* of these two circles at A to be $\pi - m\angle OAO'$. Likewise, the angle of intersection at B is $\pi - \angle OBO'$.

Remark: We could have chosen the angle between the circles to be $\angle OAO'$. The reason for our choice is that, if two circles are tangent, and each lies outside the other, then, by continuity, the angle between them is 0, while if one lies inside the other, then the angle between them is π.

Proposition 6.5. *If the circles Σ and Σ' intersect at A and B, then the angle of intersection at A has the same measure as the angle of intersection at B.*

Proof. We draw the line AB, which is a chord for both circles. We know that OO' is the perpendicular bisector of this chord; let C be the point of intersection of the lines AB and OO'.

Observe first that, by *sss*, $\triangle OAC \cong \triangle OBC$ and $\triangle O'AC \cong \triangle O'BC$. It follows that $m\angle OAC = m\angle OBC$ and that $m\angle O'AC = m\angle O'BC$. Hence $m\angle OAO' = m\angle OBO'$. □

Since the angle of intersection at A and the angle of intersection at B have the same measure, we can simply call it the *angle of intersection* of the two circles.

We remark that, since the tangent to Σ at A is orthogonal to OA, and the tangent to Σ' at A is orthogonal to $O'A$, then the angle between the lines OA and $O'A$ has the same measure as one of the angles between these tangents.

Proposition 6.6. *Suppose we are given three positive real number, a, b and c, where $a < c$, $b < c$ and $a + b > c$. Then there is a triangle with sides a, b and c.*

Proof. Consider the line segment AB, where $|AB| = c$. Draw the circle of radius a centered at B, and draw the circle of radius B centered at a. Since $a + b > c$ there are points on AB that lie inside both circles. Hence either one circle lies inside the other, or the circles intersect.

Since $a < c$, the point A lies outside the circle centered at B, and since $b < c$, the point B lies outside the circle centered at A. Hence neither circle lies inside the other, and so the circles intersect. Let C be one of the points of intersection, and observe that $|AC| = b$ and $|BC| = a$. □

6.4. Orthogonal circles. Two circles are *orthogonal* if the angle between them is $\pi/2$.

Proposition 6.7. *Let Σ be a circle with center O and radius r. Let A be some point on Σ, and let $r' > 0$ be any real number. Then there is a unique circle Σ' of radius r', orthogonal to Σ, where the center O' of Σ' lies on the line OA, and O' lies on the same side of O as does A.*

Proof. We first prove uniqueness. Suppose we have such a circle Σ'. Let A be one of the two points of intersection of Σ and Σ'. Then the triangle $OO'A$ is a right triangle with right angle at A. Hence, by the Pythagorean theorem, $|OO'|^2 = r^2 + (r')^2$. This shows that the distance from O to O' is determined; hence the circle Σ' is determined.

To prove existence, find the point O' on OA, on the same side of O as A, and at distance

$$\sqrt{r^2 + (r')^2}$$

from O. Then construct the circle Σ' of radius r' at that point.

To show that Σ and Σ' intersect, it suffices to show that there is a point B on both Σ and Σ', or equivalently, that there is a triangle with side lengths, r, r' and $|OO'|$. This follows from the above proposition, once we observe that $r < \sqrt{r^2 + (r')^2}$, $r' < \sqrt{r^2 + (r')^2}$, and $\sqrt{r^2 + (r')^2} < r + r'$. □

We remark without proof that, given two orthogonal circles Σ and Σ', there is a 1-parameter family of circles orthogonal to both Σ and Σ'. However, given three mutually orthogonal circles, there is no fourth circle orthogonal to all three.

6.5. Tangent circles.

Proposition 6.8. *Let Σ be a given circle of radius r and center O. Let A be any point, where $A \neq O$ and A does not lie on Σ. Then there is a circle Σ', centered at A, where Σ' and Σ are tangent.*

Proof. Draw the line OA. This line intersects Σ in two points; let B be one of them. Draw the circle Σ' of radius $|AB|$ about A. This circle certainly meets Σ at B. Since B lies on the line connecting the centers of the circles, the circles are tangent at B. □

We remark that we have in fact shown that there are exactly two circles centered at A that are tangent to Σ.

There are three possible orientations for the two tangent circle. We can have that Σ lies outside Σ' and Σ' lies outside Σ, or we can have that Σ lies inside Σ', or we can have that Σ' lies inside Σ.

Exercise: Suppose A, B and C are three given points on a line. How many distinct triples of mutually tangent circles are there, where one of the circles is centered at A, one is centered at B, and the third is centered at C.

Proposition 6.9. *Let $\triangle ABC$ be given. Then there are three mutually tangent circles, Σ_A centered at A, Σ_B centered at B, and Σ_C centered at C. If we require that each of the three circles lies outside the others, then the radii of these circles are determined by the lengths of the sides of the triangle.*

Proof. We need to find the radii; call these α, β and γ, where α is the radius of Σ_A, β is the radius of Σ_B and γ is the radius of Σ_C. Then we must solve the equations:

$$\alpha + \beta = |AB|, \quad \beta + \gamma = |BC|, \quad \gamma + \alpha = |CA|.$$

It is an exercise in linear algebra to show that these equations have a unique solution. □

We close with the remark that, given three mutually tangent circles, there exist exactly two disjoint circles that are tangent to all three. If one has four mutually tangent circles, then there can be no fifth.