5. Circles and lines

5.1. Circles. A circle Σ is the set of points at fixed distance $r > 0$ from a given point, its center. The distance r is called the radius of the circle Σ.

The circle Σ divides the plane into two regions: the inside, which is the set of points at distance less than r from the center O, and the outside, which consists of all points having distance from O greater than r. Note that every line segment from O to a point on Σ has the same length r.

A line segment from O to a point on Σ is also called a radius; this should cause no confusion.

A line segment connecting two points of Σ is called a chord, if the chord passes through the center, then it is called a diameter.

As above, we also use the word diameter to denote the length of a diameter of Σ, that is, the number that is twice the radius.

Proposition 5.1. *A line k intersects a circle Σ in at most two points.*

Proof. Suppose we had three points, A, B and C, of intersection of k with Σ.

We first take up the case that k is a diameter. In this case, we would have at least two of the three points on the same side of O on k; hence we can suppose that A and B both lie on the same side of O. However, by the ruler axiom (Axiom 6), we must have $|OA| \neq |OB|$ since $A \neq B$. This contradicts our assumption that A and B both lie on Σ.

We next take up the case that k is not a diameter. We can assume that B lies between A and C on k. Draw the line segments, OA, OB, OC. Then OAB, OAC and OBC are triangles. In fact, since $|OA| = |OB| = |OC|$, they are isosceles triangles. Let α be the measure of the base angles of triangle OAB. Then it is also the measure of the base angle of $\triangle OAC$, and so it is also the measure of the base angle of $\triangle OBC$. Since the two base angles at B add up to π, we obtain that each of the three triangles have two right angles, which is impossible. □

Proposition 5.2. *Let AB be a chord of a circle Σ with center O. Then the perpendicular bisector of AB passes through O.*

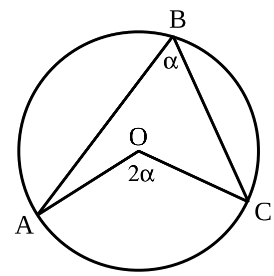
Exercise 5.1: Prove the preceding proposition.

5.2. Central angles. While we have spent a fair amount of time determining when two angles have the same measure, we have not discussed explicitly calculating the measure of an angle, except in the case of an angle of measure π and a right angle (measure $\pi/2$). We shall do so now.

First, we assume the well-known property that the circumference (that is, the arc length) of a circle of radius r is $2\pi r$.

Now let A and B be two points on a circle of radius 1 and center O. The radii AO and BO make two angles AOB (the "inner" and the "outer" angles); call them α and β. An angle such as α whose vertex is at the center of the circle is called a central angle. Notice that $\alpha + \beta = 2\pi$, no matter where A and B lie on Σ. If A and B are the endpoints of a diameter, they divide the circle into two arcs, each of length π; note also that the measure of the angles α and β are also π. In other cases, the length of the arc subtended by the angle α will be whatever fraction of 2π that α is of the entire circle. For example, if α is a right angle, it will take up $1/4$ of the circle, and the corresponding arc length will be $\pi/2$. We define the measure of the angle to be the corresponding arc length when that angle is the central angle of a circle of radius 1.

Theorem 5.3 (measure of inscribed angle is half the central angle)**.** *Let A, B and C be points on the circle Σ of radius r. Draw the chords AB and BC, and draw the radii, OA and OC. Let α be the measure of the inscribed angle ABC. Then the measure of the central angle AOC is 2α. (Here we mean the angle AOC which subtends the arc not containing B.)*

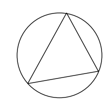


Proof. Draw the line OB. This divides quadrilateral $ABCO$ into two isosceles triangles. Let β be the measure of the base angles of $\triangle OAB$, and let γ be the measure of the base angles of $\triangle OBC$ Then the measure of the requisite central angle is given by

$$m\angle AOC = 2\pi - (\pi - 2\beta) - (\pi - 2\gamma) = 2(\beta + \gamma) = 2m\angle ABC = 2\alpha$$

□

5.3. Circumscribed circles. The circle Σ is circumscribed about $\triangle ABC$ if all three vertices of the triangle lie on the circle. In this case, we also say that the triangle is inscribed in the circle.



Note that another way to describe a circle circumscribed about a triangle is to say that it is the smallest circle for which every point inside the triangle is also inside the circle. In this view, the problem of circumscribing a circle becomes a minimization problem. A given triangle lies inside many circles, but the circumscribed circle is, in some sense, the smallest circle which lies outside the given triangle.

It is not immediately obvious that one can always solve this minimization problem, nor that the solution is unique.

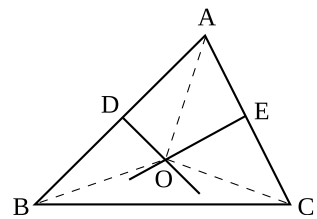
Proposition 5.4 (Uniqueness of Circumscribed Circles)**.** *There is at most one circle circumscribed about any triangle.*

Proof. Suppose there are two circles Σ and Σ' which are circumscribed about $\triangle ABC$. Since points A, B, and C lie on both circles, AB and BC are chords. By Prop. 5.2, the perpendicular bisectors of AB and BC both pass through the centers of Σ and Σ'. Since these two distinct lines can intersect in at most one point, Σ and Σ' share the same center O. Since AO is a radius for both circles, they have the same center and radius, and hence are the same circle. □

Theorem 5.5 (Existence of Circumscribed Circles)**.** *Given $\triangle ABC$, there is always exactly one circle Σ circumscribed about it.*

Proof. We need to show that the perpendicular bisectors of the sides of $\triangle ABC$ meet at a point, and that this point is equidistant from all three vertices. Then the requisite circle will have this point as its center O, and the radius will be the length of AO. Uniqueness was shown in Prop. 5.4.

Let D and E be the midpoints of sides AB and BC respectively. Draw the perpendicular bisectors of AB and BC, and let O be the point where these two lines meet (note that O need not be inside the triangle). Draw the lines AO, BO and CO.



We cannot have both that $O = D$ and $O = E$ (since $D \neq E$), hence we can assume without loss of generality that $O \neq D$. Then we have $|AD| = |DB|$, angles $\angle ADO$ and $\angle BDO$ are both right angles, and of course, $|DO| = |DO|$. Hence, $\triangle ADO \cong \triangle BDO$ by SAS. In particular, $|AO| = |BO|$. If $O = E$, then we have shown that $|AO| = |BO| = |CO|$, from which it follows that there is a circumscribed circle with center O and radius $|AO|$.

If $O \neq E$, then we repeat the above argument to show that $\triangle BOE \cong \triangle COE$, from which, as above, it follows that $|OB| = |OC|$. Again, this shows that there is a circumscribed circle. □

Corollary 5.6. *In any triangle, the three perpendicular bisectors of the sides meet at a point.*

Exercise 5.2: Explain why Theorem 5.5 implies this corollary.

Corollary 5.7 (Three Points Determine a Circle)**.** *Given any three non-colinear points, there is exactly one circle which passes through all three of them.*

Exercise 5.3: Explain why this corollary follows from Theorem 5.5.

5.4. Tangent lines and inscribed circles. A line that meets a circle in exactly one point is a tangent line to the circle at the point of intersection. Our first problem is to show that there is one and only one tangent line at each point of a circle.

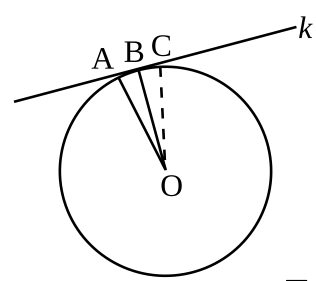
Proposition 5.8. *Let A be a point on the circle Σ, and let k be the line through A perpendicular to the radius at A. Then k is tangent to Σ.*

Proof. There are only three possibilities for k: it either is disjoint from Σ, which cannot be, as A is a common point; or it is tangent to Σ at A; or it meets Σ at another point B. If k meets Σ at B then OAB is a triangle, where $\angle A$ is a right angle. Since OA and OB are both radii, $|OA| = |OB|$. Hence $\triangle OAB$ is isosceles. Hence $m\angle A = m\angle B$. We have constructed a triangle with two right angles, which cannot be; i.e., we have reached a contradiction. □

Proposition 5.9. *If k is a line tangent to the circle Σ at the point A, then k is perpendicular to the radius ending at A.*

Proof. We will prove the contrapositive: if k is a line passing through A, where k is not perpendicular to the radius, then k is not tangent to Σ.

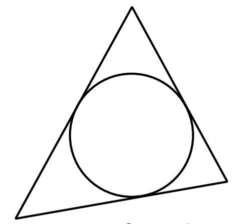
Draw the line segment m from O to k, where m is perpendicular to k. Let B be the point of intersection of k and m. On k, mark off the distance $|AB|$ from B to some point C, on the other side of B from A. Since OB is perpendicular to k, $m\angle OBA = m\angle OBC$. By SAS, $\triangle OBA \cong \triangle OBC$, and so $|OC| = |OA|$. Thus both A and C lie on Σ, and k intersects Σ in two points. Thus, k is not tangent to Σ.



□

Corollary 5.10. *Let A be a point on the circle Σ. Then there is exactly one line through A tangent to Σ.*

Exercise 5.4: Prove this Corollary.



A circle Σ is inscribed in $\triangle ABC$ if all three sides of the triangle are tangent to Σ. One can view the inscribed circle as being the largest circle whose interior lies entirely inside the triangle. (Note that it is not quite correct to say that the circle lies entirely inside the triangle, because the triangle and the circle share three points.)

We start the search for the inscribed circle with the question of what it means for the circle to have two tangents which are not parallel.

Proposition 5.11. *Let A be a point outside the circle Σ, and let k_1 and k_2 be tangents to Σ emanating from A. Then the line segment OA bisects the angle between k_1 and k_2.*

Proof. Let B_i be the point where k_i is tangent to Σ, for $i = 1, 2$. Draw the lines OB_1 and OB_2. Observe that $|OB_1| = |OB_2|$, and that, since radii are perpendicular to tangents, $\angle OB_1A = \angle OB_2A$, and these are both right angles.

By SSA, $\triangle OB_1A \cong \triangle OB_2A$. Hence $m\angle OAB_1 = m\angle OAB_2$. □

From the above, we see that if there is an inscribed circle for $\triangle ABC$, then its center lies at the point of intersection of the three angle bisectors, and its radius is the distance from this point to the three sides. Hence we have proven the following.

Corollary 5.12 (Inscribed circles are unique)**.** *Every triangle has at most one inscribed circle.*

Theorem 5.13. *Every triangle has an inscribed circle.*

Proof. Let G be the point of intersection of the angle bisectors from A and B in $\triangle ABC$. Let D be the point where the orthogonal from G meets AB; let E be the point where the orthogonal from G meets BC; and let F be the point where the orthogonal from G meets AC.

Observe that, by AAS, $\triangle ADG \cong \triangle AFG$. Similarly, $\triangle BDG \cong \triangle BEG$ and $\triangle CEG \cong \triangle CFG$.

We have shown that the perpendiculars from G to the three sides all have equal length; call this length r. then the circle centered at G of radius r is tangent to the three sides of $\triangle ABC$ exactly at the points D, E and F. □

This theorem gives another proof of the result of exercise 2.16.

Corollary 5.14. *The three angle bisectors of a triangle meet at a point; this point is the center of the inscribed circle.*

Exercise 5.5: Give a proof of this corollary using the above theorem.

Exercise 5.6: Let $\triangle ABC$ and $\triangle A'B'C'$ be such that $|AB| = |A'B'|$, $|BC| = |B'C'|$, and $m\angle C = m\angle C' = \pi/2$. Prove that $\triangle ABC \cong \triangle A'B'C'$.

Exercise 5.7: Let A and B be points on the circle Σ. Let k be the line tangent to Σ at A and let m be the line tangent to Σ at B. Prove that if k and m are parallel, then the line segment AB is a diameter of Σ.