MATH 141 Solutions to Midterm 1

8 pts. 1. (a) Give a complete and careful definition of the statement
"the sequence $\{a_k\}_{k=0}^{\infty}$ is bounded."

Solution: If $\{a_k\}$ is bounded, there is a real number M so that $|a_k| < M$ for all k.

8 pts. (b) Give a complete and careful definition of the following statement:
"The sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ converges to the limit L."

Solution: The sequence $\{a_n\}_{n=1}^{\infty}$ converges to the limit L if, for every $\epsilon > 0$, there is an integer K so that $|a_n - L| < \epsilon$ for every $n > K$.

8 pts. (c) State the Completeness Axiom for $\mathbb{R}$.

Solution: Every bounded monotone sequence is convergent.

8 pts. (d) Give a complete and careful definition of the statement
"the number L is an accumulation point of the sequence $\{x_n\}_{n=0}^{\infty}$."

Solution: If L is an accumulation point of $\{x_n\}_{n=0}^{\infty}$, then there is a subsequence $\{y_k\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ for which y_k converges to L.

20 pts. 2. Show that the function $g(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x & \text{for } x \leq 0 \end{cases}$ is continuous for all real numbers x.

Solution: Since $g(x) = x^2$ for $x \in (0, \infty)$ and x^2 is continuous, $g(x)$ is continuous for $x > 0$. Similarly, since $g(x) = -x$ on $(-\infty, 0)$, we have $g(x)$ continuous for $x < 0$.

Now we need to worry about $g(x)$ near $x = 0$. We must show that for any sequence $\{x_i\}$ which converges to 0, we have $g(x_i) \to g(0) = 0$. Since $x_i \to 0$, we may restrict our attention to values of x_i near zero, specifically $|x_i| < 1$. For such x_i, we have $|x_i|^2 < |x_i|$. In particular, if $0 < x_i < 1$, we have $g(x_i) = x_i^2 < x_i$, and for $x_i < 0$, $g(x_i) = -x_i = |x_i|$. This means that for any x_i near 0, we have

$$|g(x_i)| \leq |x_i|.$$

Now, let $\epsilon > 0$. Since $x_i \to 0$, we know there is a K so that $|x_i| < \epsilon$ for all $i > K$. Thus, we have

$$|g(x_i) - g(0)| = |g(x_i) - 0| = |g(x_i)| \leq |x_i| < \epsilon$$

for all $i > K$, and so $g(x_i) \to 0$.

Since $g(x_i)$ converges to $g(0)$ whenever x_i converges to 0, the function g is continuous at 0. Combining this with the first paragraph gives us continuity of g for all real numbers.

3. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively by $\quad a_1 = 1, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$

15 pts.

(a) Use induction to show that for all n, we have $1 \leq a_n \leq 2$

Solution: For the base case, we have that $a_1 = 1$, and certainly $1 \leq 1 \leq 2$ holds.
For the inductive step, we must show that whenever $1 \leq a_n \leq 2$, we also have $1 \leq a_{n+1} \leq 2$.
But

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \frac{1}{2}\left(1 + \frac{2}{2}\right) = 1,$$

(using the fact that $1 \leq a_n \leq 2$).
Furthermore,

$$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \leq \frac{1}{2}\left(2 + \frac{2}{1}\right) = 2,$$

which gives us $1 \leq a_{n+1} \leq 2$, as desired.
Since we have established that the property holds for a_1 and whenever it holds for a_n, it must also hold for a_{n+1}, induction tells us it must be true for all $n \geq 1$.

15 pts.

(b) Show that for $n > 2$, the sequence $\{a_n\}$ is decreasing. (Hint: look at a_{n+1}/a_n.)

Solution: If $a_{n+1}/a_n < 1$, then we know the sequence is decreasing. Observe that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2}(a_n + 2/a_n)}{a_n} = \frac{1}{2} + \frac{1}{a_n^2},$$

which will be less than 1 whenever $a_n > \sqrt{2}$.
However, we still must establish that if $a_n > \sqrt{2}$, we know that $a_{n+1} > \sqrt{2}$ (that is, once it is larger than $\sqrt{2}$, it stays bigger). To see this, write $a_n = \sqrt{2} + x$ where $x > 0$. Then we have

$$\begin{aligned} a_{n+1} = \frac{1}{2}\left(\sqrt{2} + x + \frac{2}{\sqrt{2}+x}\right) &= \frac{1}{2}\left(\sqrt{2} + x + \frac{2(\sqrt{2}-x)}{2-x^2}\right) \\ &> \frac{1}{2}\left(\sqrt{2} + x + \frac{2(\sqrt{2}-x)}{2}\right) = \frac{1}{2}\left(\sqrt{2}+\sqrt{2}\right) = \sqrt{2}. \end{aligned}$$

and so whenever $a_n > \sqrt{2}$, we know $a_{n+1} > \sqrt{2}$.
Since $a_2 = 3/2 > \sqrt{2}$, we know $a_n > \sqrt{2}$ for all $n \geq 2$, and so $a_n > a_{n+1}$ for $n \geq 2$.

5 pts.

(c) Does the sequence converge? Justify your answer.

Solution: Since the sequence is bounded (by part a), and it is decreasing after the second term (by part b), it must converge by the completeness axiom.
In fact, it converges to $\sqrt{2}$, but I didn't ask you to do that.

4. For each of series below, determine if it converges. If it converges, give the limit and a brief justification. If it fails to converge, write that it diverges and give a justification.

10 pts. (a) $\displaystyle\sum_{j=1}^{\infty} \frac{\pi^{j+2}}{5^j}$

Solution: This converges; it is a geometric series with ratio $\pi/5$ and a first term of $\pi^3/5$. This means we can reindex the sum (letting $k = j - 1$) as

$$\sum_{j=1}^{\infty} \frac{\pi^{j+2}}{5^j} = \sum_{k=0}^{\infty} \frac{\pi^{k+3}}{5^{k+1}} = \frac{\pi^3}{5} \sum_{k=0}^{\infty} \left(\frac{\pi}{5}\right)^k = \frac{\pi^3/5}{1 - \pi/5} = \frac{\pi^3}{5 - \pi},$$

(using the formula for the sum of a geometric series).

If you didn't like that way, you can also get the same answer as follows:

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{\pi^{j+2}}{5^j} = \pi^2 \sum_{j=1}^{\infty} \left(\frac{\pi}{5}\right)^j &= \pi^2 \left(-1 + \sum_{j=0}^{\infty} \left(\frac{\pi}{5}\right)^j\right) \\
&= -\pi^2 + \frac{\pi^2}{1 - \pi/5} \\
&= \frac{-(1 - \pi/5)\pi^2 + \pi^2}{1 - \pi/5} = \frac{\pi^3/5}{1 - \pi/5} = \frac{\pi^3}{5 - \pi}.
\end{aligned}$$

10 pts. (b) $\displaystyle\sum_{k=0}^{\infty} \frac{-k^3}{k^4 - \pi^4}$

Solution: This diverges.

First, write $\displaystyle\sum_{k=0}^{\infty} \frac{-k^3}{k^4 - \pi^4} = -\sum_{k=0}^{\infty} \frac{k^3}{k^4 - \pi^4}$.

Then observe that for $k \geq 3$ we have $\displaystyle\frac{k^3}{k^4 - \pi^4} > \frac{k^3}{k^4} = \frac{1}{k}$.

Since the harmonic series $\sum \frac{1}{k}$ diverges to $+\infty$, we can apply the comparison test to see that $\displaystyle\sum_{k=3}^{\infty} \frac{k^3}{k^4 - \pi^4}$ diverges, and consequently so does the original sum.