1. (3 points each)

(a) Let \( a \) be a point in the domain of a function \( f \) that is not an isolated point. Define precisely what it means for \( f \) to be continuous at \( a \).

\[
f \text{ is continuous at } a \text{ if, for every sequence } \{x_n\} \text{ in the domain of } f \text{ such that } x_n \to a, \text{ it is also true that } f(x_n) \to f(a).
\]

(b) Let \( S \) be a subset of \( \mathbb{R} \). Define \( \sup S \) and \( \inf S \).

\[
\sup S \text{ is the least upper bound for } S; \text{ that is, it is an upper bound for } S (\sup S \geq x \text{ for all } x \in S), \text{ and if } B \text{ is any upper bound for } S, \text{ then } \sup S \leq B
\]

\[
\inf S \text{ is the greatest lower bound for } S; \text{ that is, it is a lower bound for } S (\inf S \leq x \text{ for all } x \in S), \text{ and if } b \text{ is any lower bound for } S, \text{ then } \inf S \geq b
\]

(c) State the Intermediate Value Theorem.

If \( f \) is continuous on \([a, b]\) and \( C \) is any value between \( f(a) \) and \( f(b) \), then there exists \( t \in [a, b] \) such that \( f(t) = C \).

(d) State the Extreme Value Theorem.

If \( f \) is continuous on \([a, b]\), then it is bounded on this interval, and it attains its minimum and maximum values; that is, there exist \( t_{\min} \) and \( t_{\max} \) such that \( \inf f = f(t_{\min}) \) and \( \sup f = f(t_{\max}) \).

(e) State the Squeeze Theorem for functions.

Suppose \( f, g, h \) have the same domain and \( a \) is an accumulation point of their common domains, and suppose further that \( f(x) \leq h(x) \leq g(x) \) for all \( x \) in their domain. If

\[
\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L,
\]

then \( \lim_{x \to a} h(x) = L \).

(f) Give the definition of the derivative \( f'(a) \) of a function \( f \) at a point \( a \) in its domain.

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

Either definition is acceptable.
2. Find the value of each of the following limits, if it exists; otherwise, write D.N.E. (does not exist). (3 points each)

(a) \[ \lim_{x \to -1} \frac{x + 1}{x^2 - 1} \]

\[ = \lim_{x \to -1} \frac{x + 1}{(x + 1)(x - 1)} \]

\[ = \lim_{x \to -1} \frac{1}{x - 1} \quad \text{because } x \neq -1 \text{ when computing the limit} \]

\[ = \frac{1}{-2} = -\frac{1}{2} \]

(b) \[ \lim_{x \to 0} \frac{x}{\sin x} \]

\[ = \lim_{x \to 0} \frac{1}{(\sin x)/x} \]

\[ = \frac{1}{1} \quad \text{because } \frac{\sin x}{x} \to 1 \text{ as } x \to 0 \]

\[ = 1 \]

(c) \[ \lim_{x \to 0} \frac{x}{\cos x} \]

\[ = \lim_{x \to 0} \frac{x}{\cos x} = \frac{0}{1} = 0 \]

(d) \[ \lim_{x \to \infty} \frac{1}{x} \]

\[ = \lim_{y \to 0} \tan y \quad \text{because } \frac{1}{x} \to 0 \text{ as } x \to \infty \]

\[ = \frac{\lim_{y \to 0} \sin y}{\lim_{y \to 0} \cos y} = \frac{0}{1} = 0 \]

(e) \[ \lim_{x \to \infty} \tan^{-1} x \]

\[ = \frac{\pi}{2} \quad \text{by definition of the inverse tangent function} \]

(Partial credit will be given to answers based on the assumption that \( \tan^{-1} x = 1/\tan x \), despite several mentions in class that the convention \( \tan^n x = (\tan x)^n \) only applies to positive values of \( n \).)

(f) \[ \lim_{x \to 1} \ln |x - 1| \]

\[ = \lim_{y \to 0} \ln |y| \quad \text{D.N.E. because the function diverges to } -\infty \]
3. Compute the following derivatives. (3 points each)

(a) $f'(3)$, where $f(x) = 3x^3 - 2x^2 + x - 1$

$$f'(3) = \frac{d}{dx}(3x^3 - 2x^2 + x - 1) \bigg|_{x=3}$$
$$= (9x^2 - 4x + 1) \bigg|_{x=3}$$
$$= 81 - 12 + 1 = 70$$

(b) $g'(2)$, where $g(x) = \tan^{-1} x$

$$g'(2) = \frac{d}{dx} \tan^{-1} x \bigg|_{x=2}$$
$$= \frac{1}{1 + x^2} \bigg|_{x=2}$$
$$= \frac{1}{1 + 2^2} = \frac{1}{5}$$

(c) $\frac{d}{dx}((x + \cos x)e^x)$

$$= (x + \cos x)e^x + (1 - \sin x)e^x = (\cos x - \sin x + x + 1)e^x$$

(d) $\frac{d}{dx} \ln(1 + x^2)$

$$\frac{2x}{1 + x^2}$$

(e) $\frac{d}{dx} \left( \frac{\sin(x^3)}{1 + e^x} \right)$

$$= \frac{(1 + e^x)(\cos x^3)(3x^2) - (\sin x^3)e^x}{(1 + e^x)^2}$$

(f) $\frac{d}{dx} \sin((x + 1)^2(x + 2))$

$$\cos((x + 1)^2(x + 2)) \frac{d}{dx} (x + 1)^2(x + 2)$$
$$= \cos((x + 1)^2(x + 2))(2(x + 1)(x + 2) + (x + 1)^2)$$
4. (a) Recall that the hyperbolic sine and cosine functions are defined by
\[ \sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}. \]

Show that \( \frac{d}{dx} \sinh x = \cosh x \) and \( \frac{d}{dx} \cosh x = \sinh x \). (Note: unlike the case of the trigonometric functions, the signs of these do not change when taking derivatives.) (6 points)

Using the Chain Rule and the Linearity Properties of derivatives, we find
\[ \frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \]
and
\[ \frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x \]
which is what we wanted to show.

(b) Use part (a) and the relation \( \cosh^2 x - \sinh^2 x = 1 \) to find the derivative of \( \sinh^{-1} x \), the inverse hyperbolic sine. (Use the Inverse Function Rule.) (8 points)

Set \( y = \sinh^{-1} x \), so that \( x = \sinh y \). By the Inverse Function Rule,
\[ \frac{d}{dx} \sinh^{-1} x = \frac{1}{\frac{d}{dy} \sinh y} \]
\[ = \frac{1}{\cosh y} \]
\[ = \frac{1}{\sqrt{1 + \sinh^2 y}} \]
\[ = \frac{1}{\sqrt{1 + x^2}}. \]

(Note: \( \sinh x \) is strictly increasing on all of \( \mathbb{R} \), and it takes all real values, thus its inverse is also defined on all of \( \mathbb{R} \). The formula above shows that its inverse is differentiable on all of \( \mathbb{R} \) and that the derivative is continuous.)

The graphs of \( y = \sinh x \) and \( y = \sinh^{-1} x \).
5. Let \( p(x) = ax^3 + bx^2 + cx + d \) be a cubic polynomial with \( a > 0 \) and \( d < 0 \).

(a) Show that \( \lim_{x \to \infty} \frac{p(x)}{x^3} = a \). (6 points)

\[
\lim_{x \to \infty} \frac{p(x)}{x^3} = \lim_{x \to \infty} \frac{ax^3 + bx^2 + cx + d}{x^3} \\
= \lim_{x \to \infty} \left( a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} \right) \\
= a + 0 + 0 + 0 = a
\]

(b) Use part (a) to show that \( p(x) > 0 \) for some \( x > 0 \). (6 points)

Because \( \lim_{x \to \infty} \frac{p(x)}{x^3} = a \) and \( a > 0 \), for some \( x > 0 \) we must have \( \frac{p(x)}{x^3} > \frac{a}{2} \). This implies \( p(x) > \frac{a x^3}{2} \). Now the inequalities \( a > 0 \) and \( x > 0 \) imply that \( p(x) > 0 \).

(c) Use part (b) and the Intermediate Value Theorem to show that \( p(x) \) equals zero for some \( x > 0 \). (6 points)

Let \( x_0 \) be the point found in part (b). Because \( p(x) \) is continuous on all of \( \mathbb{R} \), it is in particular continuous on \([0, x_0]\). Because \( d < 0 \), we have \( p(0) = d < 0 \); by our choice of \( x_0 \) we also have \( p(x_0) > d \). Therefore \( p(0) < 0 < p(x_0) \), and by the Intermediate Value Theorem, there exists \( x \in [0, x_0] \) such that \( p(x) = 0 \).
6. (a) Let \( f \) and \( g \) be functions defined on all of \( \mathbb{R} \). Suppose that \( f \) is strictly increasing and \( g \) is strictly decreasing. Show that there is at most one point of \( \mathbb{R} \) where \( f \) and \( g \) are equal. Do not assume that either function is differentiable. (Hint: What happens if you assume that \( f \) and \( g \) are equal at two distinct points of \( \mathbb{R} \)?) (8 points)

Suppose, by way of contradiction, that \( f \) and \( g \) were equal at two distinct points, say \( x \) and \( y \) with \( x < y \). Because \( f \) is strictly increasing, \( f(x) < f(y) \), and because \( g \) is strictly decreasing, \( g(x) > g(y) \). But \( f(x) \) and \( g(x) \) are equal, so we have

\[
g(y) < g(x) = f(x) < f(y).
\]

Now we see that \( g(y) \) and \( f(y) \) cannot be equal, because no number can be strictly greater than itself. This is a contradiction to our choice of \( y \), and so \( f \) and \( g \) cannot be equal at two distinct points, which is the same thing as saying there is at most one point where they are equal.

(b) Give an example of a pair of continuous functions \( f \) and \( g \) defined on all of \( \mathbb{R} \) such that \( f \) is strictly increasing, \( g \) is strictly decreasing, and \( f \) and \( g \) are never equal. (6 points)

An obvious example is \( f(x) = e^x \), \( g(x) = -e^x \). Other examples are possible.