MATH 141 Solutions to Midterm 1

5 pts. 1. (a) Give a complete and careful definition of the following statement:
"The sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ converges to the limit L."

Solution: The sequence $\{a_n\}_{n=1}^{\infty}$ converges to the limit L if, for every $\epsilon > 0$, there is an integer K so that $|a_n - L| < \epsilon$ for every $n > K$.

5 pts. (b) Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$. Give a complete definition of the following statement:
"The function $f(x)$ is continuous at $x = a$."

Solution: $f(x)$ is continuous at $x = a$ if $f(a)$ is defined, and for every sequence x_n which converges to a, the sequence $f(x_n)$ converges to $f(a)$.

We could also just say $f(x)$ is continuous at $x = a$ if $f(a)$ is defined, and $\lim_{x\to a} f(x) = a$, which is exactly the same thing. The first version just avoids the limit.

If you prefer, you could also use the ϵ-δ definition: $f(x)$ is continuous at $x = a$ if $f(a)$ is defined, and for every $\epsilon > 0$, there is δ so that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

5 pts. (c) If A is a subset of the real numbers, give a complete and careful definition of the following statement: $\sup A = \alpha$.

Solution: The supremum of a set of reals A is α if the following holds:

- α is an upper bound for A. That is, for every element $x \in A$, we have $x \le \alpha$.
- If β is also an upper bound for A, we must have $\alpha \le \beta$.

5 pts. (d) State the Intermediate Value Theorem for a function $f : \mathbb{R} \to \mathbb{R}$.

Solution: If $f : \mathbb{R} \to \mathbb{R}$ is continuous on an interval $[a, b]$ and y is any number between $f(a)$ and $f(b)$, then there is a number c between a and b so that $f(c) = y$.

2. Consider the sequence

$$\{r_n\} = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots$$

5 pts. (a) If this sequence has any accumulation points, list one of them. If there are none, write "none". In either case, give some justification of your answer (it needn't be a proof, just a reason why your answer is reasonable.)

Solution: There are, of course, many correct answers (see below for why). However, 0 is as good an answer as any other. This is because there is a subsequence converging to 0, for example

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \to 0.$$

10 pts. (b) Describe *all* accumulation points of the sequence above. As before, if there are none, just write "none". Give a justification of your answer.

Solution: Note that every rational number $\frac{p}{q}$ with $0 < p < q$ appears in the sequence (that is, all rationals between 0 and 1). Any real number x with $0 \leq x \leq 1$ is an accumulation point of this sequence.

One way to see this is as follows. Write the decimal expansion of x as $x = 0.a_1a_2a_3a_4\ldots$ If x is not a terminating decimal (that is, for any N, there is a $k > N$ so that $a_k \neq 0$), we may select a subsequence with the n^{th} term given by

$$x_n = \frac{a_1a_2a_3\ldots a_n}{10^n}$$

The sequence $\{x_n\}$ will converge to x.

If x has a terminating decimal expansion (that is, it ends in all zeros), there is another representation which ends in repeating 9s. Using this representation will solve the problem. Well, except for $x = 0$. In this case, we can use the sequence given in the first part of the problem.

Finally, observe that $\{r_n\}$ can have no accumulation points outside of $[0, 1]$, since no sequence of numbers between 0 and 1 can converge to a number larger than 1 or less than 0.

15 pts. 3. Prove that for any integer $n \geq 1$, $1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = (1 + 2 + 3 + 4 + \ldots + n)^2$.

The formulae $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ might or might not be helpful. Induction could be your friend, too.

Solution: To prove this by induction, we first observe that the statement holds trivially for $n = 1$, since $1^3 = 1^2$.

For the inductive step, we must show that if

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + (n-1)^3 = (1 + 2 + 3 + 4 + \ldots + (n-1))^2$$

then we have

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = (1 + 2 + 3 + 4 + \ldots + n)^2.$$

Applying the inductive hypothesis and the given formula yields

$$\begin{aligned} 1^3 + 2^3 + \ldots + (n-1)^3 + n^3 &= \big(1 + 2 + \ldots + (n-1)\big)^2 + n^3 \\ &= \left(\frac{(n-1)n}{2}\right)^2 + n^3 \\ &= \frac{n^4 - 2n^3 + n^2}{4} + \frac{4n^3}{4} \\ &= \frac{n^4 + 2n^3 + n^2}{4} \qquad = \frac{n^2(n+1)^2}{4} \end{aligned}$$

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Solution (continued): So we have shown that

$$1^3 + 2^3 + \ldots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2 = \big(1 + 2 + \ldots + (n-1) + n\big)^2$$

Since we have established that the formula holds for $n = 1$ and that if the formula holds for $n-1$, it must also hold for n, then by the principle of induction we know it must hold for all natural numbers.

15 pts. 4. (a) Let f and g be functions defined on all of $\mathbb{R}$, with f strictly increasing and g strictly decreasing. Prove that there is *at most* one point $a \in \mathbb{R}$ with $f(a) = g(a)$. Do not assume f or g are continuous.
(Hint: what goes wrong if there are two distinct points where f and g are equal?)

Solution: As the hint suggests, we prove this by contradiction. Suppose that we have $f(a) = g(a)$ and $f(b) = g(b)$ with $a < b$. Since f is strictly increasing, we have $f(a) < f(b)$, and since g is decreasing, $g(a) > g(b)$. But this means

$$g(a) = f(a) < f(b) = g(b) < g(a),$$

and so $g(a) < g(a)$, a contradiction. Thus, there is at most one point where f and g are equal.

5 pts. (b) Give an example of a pair of continuous functions f and g, defined on all of $\mathbb{R}$, with f strictly increasing and g strictly decreasing and so that f and g are never equal.

Solution: There are many such functions. One such choice is $f(x) = e^x$ and $g(x) = -e^x$.

5. For each of the sequences or series below, determine if it converges. If it converges, give the limit and a brief justification. If it fails to converge, write that it diverges and give a justification.

5 pts. (a) $\displaystyle\sum_{j=1}^{\infty} \frac{\pi}{(\ln 8)^j}$

Solution: This converges; it is a geometric series with ratio $1/\ln 8$. Since $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, the sum of this series is $\dfrac{\pi}{1 - 1/\ln 8} - 1$. (We had to subtract 1 because the term for $j = 0$ does not appear.)

5 pts. (b) $\displaystyle\sum_{j=1}^{\infty} \frac{\pi}{j - \ln 8}$

Solution: This diverges. Since for $j \geq 3$ we have $\dfrac{\pi}{j - \ln 8} > \dfrac{1}{j}$ and the harmonic series $\sum \dfrac{1}{j}$ diverges, the given series also diverges.

$\boxed{\text{5 pts.}}$ (c) $\left\{(-1)^{2n} + \dfrac{n-1}{n}\right\}_{n=1}^{\infty}$

Solution: If a_n represents the n^{th} term, we can simplify this to

$$a_n = 1 + \frac{n-1}{n},$$

which obviously converges to 1+1 = 2.

$\boxed{\text{5 pts.}}$ (d) $\{a_n\}_{n=1}^{\infty}$ where $a_1 = 1$ and $a_{k+1} = 1 + \dfrac{1}{1+a_k}$

Solution: Maybe it will help to just list a few terms first. The first several terms are

$$1,\ \frac{3}{2} = 1.5,\ \frac{7}{5} = 1.4,\ \frac{17}{12} \approx 1.41667,\ \frac{41}{29} \approx 1.41379,\ \frac{99}{70} \approx 1.41429,\ \ldots$$

So it looks like it should converge.
If a_k converges to some L, then we must have

$$L = 1 + \frac{1}{1+L} \qquad \text{or equivalently} \qquad L^2 - 1 = 1$$

This means that L, if it exists, must be either $\sqrt{2}$ or $-\sqrt{2}$. Since all terms of the sequence are positive, $L = \sqrt{2}$ is the only possibility.
However, we need to check that the sequence actually converges. To see this, one way is to show that the the odd-numbered terms are monotonically increasing and bounded above, and that the even numbered terms are monotonically decreasing and bounded below, and so both subsequences converge. Then we'll show that they converge to the same thing.
Observe that

$$a_{n+2} = 1 + \frac{1}{1+a_{n+1}} = 1 + \frac{1}{1+\frac{1}{1+a_n}} = \frac{4+3a_n}{3+2a_n}.$$

This doesn't seem to help, but it does: from this we can see that $a_{n+2} > a_n$ precisely when $a_n < \sqrt{2}$. Furthermore, from the formula if $a_n < \sqrt{2}$ then $a_{n+2} < \sqrt{2}$ and if $a_n > \sqrt{2}$, then $a_{n+2} > \sqrt{2}$.
Putting those two facts together shows that the subsequence of even terms is bounded and decreasing, and the subsequence of odd terms is bounded and increasing. Thus, both converge to some limit.
Finally, we use the same trick as before: the limit L of either subsequence must satisfy

$$L = \frac{4+3L}{3+2L}, \qquad \text{or equivalently,} \qquad L^2 = 2.$$

Hence, the limit for both subsequences is $\sqrt{2}$. Consequently, the entire sequence converges to $\sqrt{2}$.
(Naturally, I didn't expect anyone to do this in anywhere near this much detail. But I have to.)