MAT 141 Honors Calculus **Exam 2 SOLUTIONS**

1. (3 points each)

   (a) Let a be a point in the domain of a function f that is not an isolated point. Define precisely what it means for f to be continuous at a.

   f is continuous at a if, for every sequence $\{x_n\}$ in the domain of f such that $x_n \to a$, it is also true that $f(x_n) \to f(a)$.

   (b) Let S be a subset of $\mathbb{R}$. Define $\sup S$ and $\inf S$.

   $\sup S$ is the least upper bound for S; that is, it is an upper bound for S ($\sup S \geq x$ for all $x \in S$), and if B is any upper bound for S, then $\sup S \leq B$

   $\inf S$ is the greatest lower bound for S; that is, it is a lower bound for S ($\inf S \leq x$ for all $x \in S$), and if b is any lower bound for S, then $\inf S \geq b$

   (c) State the Intermediate Value Theorem.

   If f is continuous on $[a, b]$ and C is any value between $f(a)$ and $f(b)$, then there exists $t \in [a, b]$ such that $f(t) = C$.

   (d) State the Extreme Value Theorem.

   If f is continuous on $[a, b]$, then it is bounded on this interval, and it attains its minimum and maximum values; that is, there exist $t_{\min}$ and $t_{\max}$ such that $\inf f = f(t_{\min})$ and $\sup f = f(t_{\max})$.

   (e) State the Squeeze Theorem for functions.

   Suppose f, g, h have the same domain and a is an accumulation point of their common domains, and suppose further that $f(x) \leq h(x) \leq g(x)$ for all x in their domain. If

   $$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L,$$

   then $\lim_{x \to a} h(x) = L$.

   (f) Give the definition of the derivative $f'(a)$ of a function f at a point a in its domain.

   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

   Either definition is acceptable.

2. Find the value of each of the following limits, if it exists; otherwise, write **D.N.E.** (does not exist). (3 points each)

(a) $\lim\limits_{x\to -1} \dfrac{x+1}{x^2-1}$

$$\begin{aligned} &= \lim_{x\to -1} \frac{x+1}{(x+1)(x-1)} \\ &= \lim_{x\to -1} \frac{1}{x-1} && \text{because } x \neq -1 \text{ when computing the limit} \\ &= \frac{1}{-2} = \boxed{-\frac{1}{2}} \end{aligned}$$

(b) $\lim\limits_{x\to 0} \dfrac{x}{\sin x}$

$$\begin{aligned} &= \lim_{x\to 0} \frac{1}{(\sin x)/x} \\ &= \frac{1}{1} && \text{because } \frac{\sin x}{x} \to 1 \text{ as } x \to 0 \\ &= \boxed{1} \end{aligned}$$

(c) $\lim\limits_{x\to 0} \dfrac{x}{\cos x}$

$$= \frac{\lim_{x\to 0} x}{\lim_{x\to 0} \cos x} = \frac{0}{1} = \boxed{0}$$

(d) $\lim\limits_{x\to \infty} \tan \dfrac{1}{x}$

$$\begin{aligned} &= \lim_{y\to 0} \tan y && \text{because } \frac{1}{x} \to 0 \text{ as } x \to \infty \\ &= \frac{\lim_{y\to 0} \sin y}{\lim_{y\to 0} \cos y} = \boxed{0} \end{aligned}$$

(e) $\lim\limits_{x\to \infty} \tan^{-1} x$

$$= \boxed{\frac{\pi}{2}} \qquad \text{by definition of the inverse tangent function}$$

(Partial credit will be given to answers based on the assumption that $\tan^{-1} x = 1/\tan x$, despite several mentions in class that the convention $\tan^n x = (\tan x)^n$ only applies to positive values of n.)

(f) $\lim\limits_{x\to 1} \ln|x-1|$

$$= \lim_{y\to 0} \ln|y| \qquad \boxed{\text{D.N.E.}} \text{ because the function diverges to } -\infty$$

3. Compute the following derivatives. (3 points each)

(a) $f'(3)$, where $f(x) = 3x^3 - 2x^2 + x - 1$

$$\begin{aligned} f'(3) &= \frac{d}{dx}(3x^3 - 2x^2 + x - 1)\bigg|_{x=3} \\ &= (9x^2 - 4x + 1)\big|_{x=3} \\ &= 81 - 12 + 1 = \boxed{70} \end{aligned}$$

(b) $g'(2)$, where $g(x) = \tan^{-1} x$

$$\begin{aligned} g'(2) &= \frac{d}{dx}\tan^{-1} x\bigg|_{x=2} \\ &= \frac{1}{1+x^2}\bigg|_{x=2} \\ &= \frac{1}{1+2^2} = \boxed{\frac{1}{5}} \end{aligned}$$

(c) $\dfrac{d}{dx}\big((x + \cos x)e^x\big)$

$$= (x + \cos x)e^x + (1 - \sin x)e^x = \boxed{(\cos x - \sin x + x + 1)e^x}$$

(d) $\dfrac{d}{dx}\ln(1 + x^2)$

$$\boxed{\frac{2x}{1+x^2}}$$

(e) $\dfrac{d}{dx}\left(\dfrac{\sin(x^3)}{1+e^x}\right)$

$$\boxed{\frac{(1+e^x)(\cos x^3)(3x^2) - (\sin x^3)e^x}{(1+e^x)^2}}$$

(f) $\dfrac{d}{dx}\sin((x+1)^2(x+2))$

$$\begin{aligned} &\cos((x+1)^2(x+2))\frac{d}{dx}(x+1)^2(x+2) \\ &= \boxed{\cos((x+1)^2(x+2))(2(x+1)(x+2) + (x+1)^2)} \end{aligned}$$

4. (a) Recall that the hyperbolic sine and cosine functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Show that $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$. (*Note:* unlike the case of the trigonometric functions, the signs of these do *not* change when taking derivatives.) (6 points)

Using the Chain Rule and the Linearity Properties of derivatives, we find

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$\frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

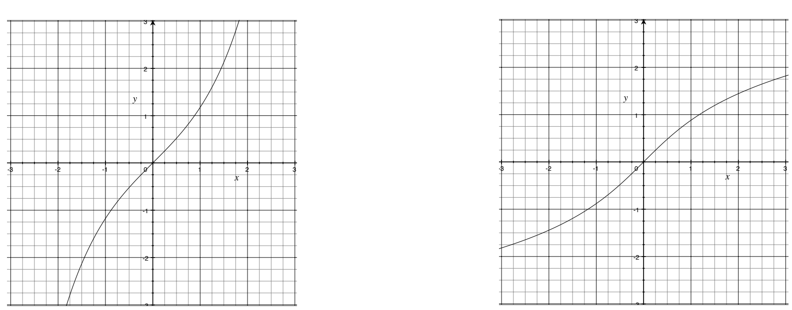
which is what we wanted to show.

(b) Use part (a) and the relation $\cosh^2 x - \sinh^2 x = 1$ to find the derivative of $\sinh^{-1} x$, the inverse hyperbolic sine. (Use the Inverse Function Rule.) (8 points)

Set $y = \sinh^{-1} x$, so that $x = \sinh y$. By the Inverse Function Rule,

$$\begin{aligned} \frac{d}{dx} \sinh^{-1} x &= \frac{1}{\frac{d}{dy} \sinh y} \\ &= \frac{1}{\cosh y} \\ &= \frac{1}{\sqrt{1 + \sinh^2 y}} \\ &= \boxed{\frac{1}{\sqrt{1 + x^2}}}. \end{aligned}$$

(*Note:* $\sinh x$ is strictly increasing on all of $\mathbb{R}$, and it takes all real values, thus its inverse is also defined on all of $\mathbb{R}$. The formula above shows that its inverse is differentiable on all of $\mathbb{R}$ and that the derivative is continuous.)



The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$.

5. Let $p(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with $a > 0$ and $d < 0$.

(a) Show that $\lim_{x\to\infty} \dfrac{p(x)}{x^3} = a$. (6 points)

$$\begin{aligned}\lim_{x\to\infty} \frac{p(x)}{x^3} &= \lim_{x\to\infty} \frac{ax^3 + bx^2 + cx + d}{x^3} \\ &= \lim_{x\to\infty} \left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right) \\ &= a + 0 + 0 + 0 = a\end{aligned}$$

(b) Use part (a) to show that $p(x) > 0$ for some $x > 0$. (6 points)

Because $\lim_{x\to\infty} \frac{p(x)}{x^3} = a$ and $a > 0$, for some $x > 0$ we must have $\frac{p(x)}{x^3} > \frac{a}{2}$. This implies $p(x) > ax^3/2$. Now the inequalities $a > 0$ and $x > 0$ imply that $p(x) > 0$.

(c) Use part (b) and the Intermediate Value Theorem to show that $p(x)$ equals zero for some $x > 0$. (6 points)

Let x_0 be the point found in part (b). Because $p(x)$ is continuous on all of $\mathbb{R}$, it is in particular continuous on $[0, x_0]$. Because $d < 0$, we have $p(0) = d < 0$; by our choice of x_0 we also have $p(x_0) > d$. Therefore $p(0) < 0 < p(x_0)$, and by the Intermediate Value Theorem, there exists $x \in [0, x_0]$ such that $p(x) = 0$.

6. (a) Let f and g be functions defined on all of $\mathbb{R}$. Suppose that f is strictly increasing and g is strictly decreasing. Show that there is at most one point of $\mathbb{R}$ where f and g are equal. Do *not* assume that either function is differentiable. (*Hint:* What happens if you assume that f and g are equal at two distinct points of $\mathbb{R}$?) (8 points)

Suppose, by way of contradiction, that f and g were equal at two distinct points, say x and y with $x < y$. Because f is strictly increasing, $f(x) < f(y)$, and because g is strictly decreasing, $g(x) > g(y)$. But $f(x)$ and $g(x)$ are equal, so we have

$$g(y) < g(x) = f(x) < f(y).$$

Now we see that $g(y)$ and $f(y)$ cannot be equal, because no number can be strictly greater than itself. This is a contradiction to our choice of y, and so f and g cannot be equal at two distinct points, which is the same thing as saying there is at most one point where they are equal.

(b) Give an example of a pair of continuous functions f and g defined on all of $\mathbb{R}$ such that f is strictly increasing, g is strictly decreasing, and f and g are *never* equal. (6 points)

An obvious example is $f(x) = e^x$, $g(x) = -e^x$. Other examples are possible.