SOLUTIONS TO MIDTERM II (MAT 132, SPRING 2010)

Each problem is worth 20 points.

(1) If 50 foot-pounds of work is needed to stretch a spring from its natural length of 10 feet to a length of 12 feet, then how much work is needed to stretch the spring from a length of 11 feet to a length of 15 feet. **Show work.**

Solution Let $x = 0$ correspond to the rest position. The force needed to overcome the force of the spring at position x is $F(x) = kx$. Thus the work needed to stretch the spring from 0 to 2 is given by $\int_0^2 kxdx = 2k = 50$; thus $k = 25$. The work needed to stretch the spring from 11 feet to 15 feet is equal to $\int_1^5 25xdx = \frac{25x^2}{2} \mid_1^5 = \frac{5^4}{2} - \frac{5^2}{2}$.

(2) Find the length of curve $y = \int_0^x (u^2 e^{u^2} - 1)^{\frac{1}{2}} du$, $1 \leq x \leq 3$. **Show work.**

Solution: Length$=\int_1^3 (1 + (y')^2)^{\frac{1}{2}} dx = \int_1^3 (1 + ((x^2 e^{x^2} - 1)^{\frac{1}{2}})^2)^{\frac{1}{2}} =$
$\int_1^3 (x^2 e^{x^2})^{\frac{1}{2}} dx = \int_1^3 x e^{\frac{x^2}{2}} dx = e^{\frac{x^2}{2}} \mid_1^3 = e^{\frac{9}{2}} - e^{\frac{1}{2}}$.

(3) Let $\mathbb{R}$ denote the region in the plane bounded by the curves $x = 2$, $x = 9$, $y = sin(x)$ and $y = 3^x$.

(a) Express the area of $\mathbb{R}$ as a definite integral.
Solution: $\int_2^9 (3^x - sin(x))dx$

(b) Let S_1 denote the solid obtained by rotating the region $\mathbb{R}$ about the line $y = -2$. Express the volume of S_1 as a definite integral.
Solution $\int_2^9 \pi((3^x + 2)^2 - (sin(x) + 2)^2)dx$

(c) Let S_2 denote the solid obtained by rotating the region $\mathbb{R}$ about the y-axis. Express the volume of S_2 as a definite integral.
Solution: $\int_2^9 2\pi x(3^x - sin(x))dx$

(4) A spherical tank of radius 3 meters is buried in the ground so that the top of the tank is 4 meters beneath ground level. The tank is filled with a liquid which has a density of 1500 $kilograms/meter^3$. How much work does it take to pump all of the liquid to ground level? (Recall that gravitational acceleration is 9.8 $meters/second^2$.) **Show work.**

Solution: First we must place a coordinate system. I choose to let x=0 correspond to the bottom of the tank, x=6 correspond to the top of the tank and x=10 correspond to ground level. (There are many ways to impose a coordinate system on the problem; they are all equally correct.)

The radius of a cross section of the spherical tank at level x_i is equal to $(6x_i - (x_i)^2)^{\frac{1}{2}}$. So the volume of the liquid in the tank between level x_i and level $x_i + \triangle x$ is approximately equal to $\pi(6x_i - (x_i)^2)\triangle x$ cubic meters; thus the mass of this liquid is approximately equal to $1500\pi(6x_i - (x_i)^2)\triangle x$ kilograms. The force needed to lift this portion of the liquid to ground level is approximately $9.8(1500)\pi(6x_i - (x_i)^2)$ $kg - m/sec^2$=newtons for a distance of approximately $10 - x_i$ meters; thus the work required to lift this portion of the liquid to ground level is approximately $(10-x_i)9.8(1500)\pi(6x_i-(x_i)^2)\triangle x$ newton-meters=joules. So the total work (in joules) is approximated by the sums of all these portions of work

$$\sum_{i=1}^{n}(10 - x_i)9.8(1500)\pi(6x_i - (x_i)^2)\triangle x.$$

Taking the limit as $n \to \infty$ we get that the total work is equal to

$$\int_0^6 (10 - x)9.8(1500)\pi(6x - x^2)dx$$

joules.

(5) Consider the plate (also called a *lamina*) which occupies the region in the plane lying between the x-axis and the graph of $y = xln(x)$, $1 \leq x \leq e$. Suppose that this plate has constant density ρ.

(a) Compute the mass of the plate (in terms of ρ). **Show work.**
Solution: mass=$\rho \int_1^e xln(x)dx = (\frac{x^2 ln(x)}{2} - \frac{x^2}{4}) \mid_1^e = (\frac{e^2}{4}) - (-\frac{1}{4})$.

(b) Let m denote the mass of the plate and let $(\bar{x}, \bar{y})$ denote the center of mass of the plate. Express each of $m\bar{x}$ and $m\bar{y}$ as a definite integral.
Solution: $m\bar{x} = \rho \int_1^e x(xln(x))dx$ and $m\bar{y} = \rho \int_1^e \frac{1}{2}(xln(x))^2 dx$.