Part 1: Do These First!

Name: ____________ **Id:** ____________

10 pts 1. Calculate the indefinite integral: $\displaystyle\int \frac{\cos x\, dx}{\sin^2 x}$.

$u = \sin x$

$du = \cos x\, dx$

$$\int u^{-2} du = -u^{-1} + C = \boxed{-\frac{1}{\sin x} + C}$$

10 pts 2. Calculate the definite integral $\displaystyle\int_0^1 xe^{-2x}\, dx$. If it does not converge, write "Diverges".

$u = x \quad dv = e^{-2x} dx$

$du = dx \quad v = -\frac{1}{2} e^{-2x}$

$$= -\frac{x}{2} e^{-2x} + \frac{1}{2}\int_0^1 e^{-2x} dx = -\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} \Big|_0^1 = \boxed{-\frac{1}{2} e^{-2} - \frac{1}{4} e^{-2} + \frac{1}{4}}$$

10 pts 3. Calculate the definite integral $\displaystyle\int_0^1 \frac{3}{x^5}\, dx$. If it does not converge, write "Diverges".

$$\int_0^1 \frac{3}{x^5} dx = -\frac{3}{4} x^{-4} \Big|_0^1 = -\frac{3}{4} - \lim_{x \to 0^+} \left(-\frac{3}{4} x^{-4}\right)$$

$\boxed{\text{DIVERGES}}$

cuz $\lim_{x \to 0^+} \frac{-3}{4} x^{-4} = -\infty$

10 pts 4. Calculate the indefinite integral: $\displaystyle\int \frac{dx}{(x+1)(x-1)}$ $= \frac{1}{2}\int \frac{-1}{x+1} + \frac{1}{x-1}\, dx$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$1 = A(x-1) + B(x+1)$

$1 = 2B \Rightarrow B = 1/2$

$1 = -2A \Rightarrow A = -1/2$

$= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$

$= \frac{1}{2} \ln\left|\frac{x-1}{x+1}\right| + C.$

$x+1 - x-1$

$= 2.$

Part 1: Do These First!

Name: ________________ **Id:** ______________

| 10 pts | 5. Find the sum: $\sum_{n=1}^{\infty} \frac{3}{5^n}$

GEOMETRIC

SINCE $n=1$, NOT $n=0$.

$$\frac{3}{1-\frac{1}{5}} - 3 = \frac{3}{4/5} - 3 = \frac{15}{4} - 3 = \boxed{\frac{3}{4}}$$

| 10 pts | 6. Write a power series for xe^{-2x}.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$xe^{-2x} = x\left(1 - (2x) + \frac{2^2x^2}{2!} - \cdots\right) = \boxed{\sum_{n=0}^{\infty} \frac{2^n x^{n+1}(-1)^n}{n!}}$$

| 10 pts | 7. Find a function $y(x)$ that solves the differential equation $\frac{dy}{dx} = \frac{x}{\cos y}$ with $y(0) = \frac{\pi}{4}$

$$\int \cos y \, dy = \int x \, dx$$

$$\sin y = \frac{1}{2}x^2 + c$$

$$y = \arcsin\left(\frac{x^2}{2} + c\right)$$

$$y(0) = \pi/4 \Rightarrow \arcsin(c) = \pi/4$$

$$\Rightarrow c = \frac{1}{\sqrt{2}}.$$

$$\boxed{y(x) = \arcsin\left(\frac{x^2}{2} + \frac{1}{\sqrt{2}}\right)}.$$

| 10 pts | 8. The series $\sum_{n=1}^{\infty} \frac{1}{3^n\sqrt{n}}$ Converges / Diverges by the what test?

circle one

Justify:

RATIO: $\lim \frac{3^n\sqrt{n}}{3^{n+1}\sqrt{n+1}} = \frac{1}{3} < 1$

OR COMPARE W/

$$\frac{1}{3^n} > \frac{1}{3^n\sqrt{n}}$$

8. RATIO. OR COMPARISON.

| 10 pts | 9. The series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Converges / Diverges by the what test?

circle one

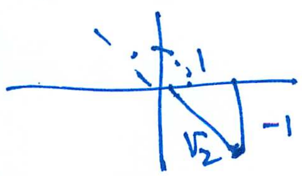
Justify:

$u = \ln n$

$$\int \frac{dx}{x(\ln n)^2} = \int_2^{\infty} \frac{du}{u^2} = \frac{-1}{u}\Big|_2^{\infty} = \frac{1}{2}$$

9. INTEGRAL

| 10 pts | 10. Write polar coordinates for the point with rectangular coordinates $(1, -1)$ in two different ways, one with $r > 0$ and the other with $r < 0$.



$r = \sqrt{2}$

$(r > 0)$ $r =$ $\sqrt{2}$ $\theta =$ $-\pi/4$

$(r < 0)$ $r =$ $-\sqrt{2}$ $\theta =$ $3\pi/4$

Part 2: Do these after part 1. **Name:** ____________ **Id:** ____________

$\boxed{15 \text{ pts}}$ 11. Write the first four nonzero terms of the Taylor series for $f(x) = \ln(x/2)$ centered at $a = 2$.

$f(x) = \ln(x/2)$ $\quad f(2) = \ln 1 = 0$

$f'(x) = \frac{2}{x}\left(\frac{1}{2}\right) = x^{-1}$ $\quad f'(2) = \frac{1}{2}$

$f''(x) = -x^{-2}$ $\quad f''(2) = -\frac{1}{4}$

$f'''(x) = 2x^{-3}$ $\quad f'''(2) = \frac{1}{4}$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(2) = -3/8$

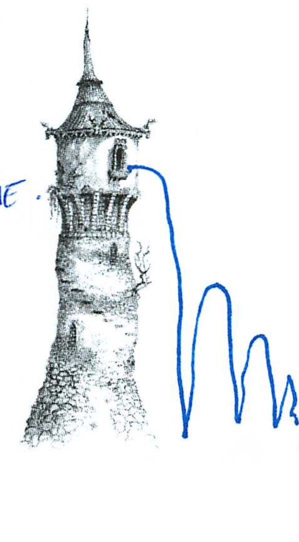
$f(x) = \frac{1}{2}(x-2) - \frac{1}{4 \cdot 2}(x-2)^2 + \frac{1}{4 \cdot 3!}(x-2)^3 - \frac{3}{8 \cdot 4!}(x-2)^4 + \cdots$

$\boxed{15 \text{ pts}}$ 12. Calculate the sum $\frac{\pi}{2} - \frac{\pi^3}{2^3\, 3!} + \frac{\pi^5}{2^5\, 5!} - \frac{\pi^7}{2^7\, 7!} + \ldots$ $= \sum_{n=1}^{\infty} (-1)\left(\frac{\pi}{2}\right)^{2n}/n!$

$= \sin\left(\frac{\pi}{2}\right)$

$= 1$

$\boxed{15 \text{ pts}}$ 13. Jill is being held prisoner by the evil monkey-king. As a signal to her brother Jack, she drops an enchanted orb out of the window of the tower where she is being held, 100 feet above the ground. Each time the orb strikes the ground, it sends out a beacon of golden light, then bounces and returns to a height two-thirds of its previous maximum height. What is the total distance traveled by the orb if it bounces infinitely many times?



GEOMETRIC. MUST COUNT BOTH BOUNCES.

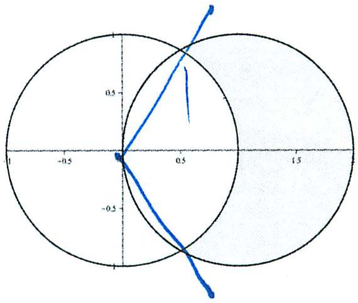
$200 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - 100$ ← DOESNT GO UP FIRST TIME.

$= \frac{200}{1 - 2/3} - 100 = 600 - 100$

$= 500'$.

Part 2: Do these after part 1. **Name:** ____________ **Id:** ____________

| 15 pts | 14. Find the area of the region that lies inside the circle of radius one given by $r = 2\cos\theta$, but outside the circle $r = 1$.



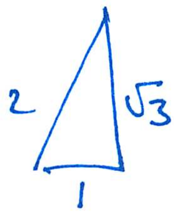
CROSS WHEN $\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \pm\frac{\pi}{3}$.

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2}(2\cos\theta)^2 - \frac{1}{2}(1)^2 \, d\theta$$

$$= 2\int_{-\pi/3}^{\pi/3}(\cos^2\theta - \int_{-\pi/3}^{\pi/3}\left(\frac{1}{2}\right)d\theta$$

$$= 2\int_{0}^{\pi/3}(1+\cos 2\theta)\,d\theta - \left(\frac{\pi}{3}\right)$$



$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right|_{0}^{\pi/3} - \pi/3$$

$$= 2\pi/3 + \frac{\sqrt{3}}{2} - \pi/3 = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

Part 2: Do these after part 1. **Name:** ______________ **Id:** ______________

15. Consider the differential equation $y'' + 4y' + 20y = 0$.

10 pts (a) Write the most general form of the solution $y(x)$ which is real-valued for x real.

$$\lambda^2 + 4\lambda + 20 = 0$$

$$\frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm 4i$$

$$\boxed{y(x) = e^{-2x}\left(A\cos(4x) + B\sin(4x)\right)}$$

$$y'(x) = -2e^{-2x}\left(A\cos 4x + B\sin 4x\right) + e^{-2x}\left(-4A\sin 4x + 4B\cos 4x\right)$$

5 pts (b) Write a formula for the solution $y(x)$ with $y(0) = 1$ and $y'(0) = 14$.

$$y(0) = 1 \Rightarrow A = 1$$

$$y'(0) = 14 \Rightarrow -2(A + 4B = 14$$

$$4B = 16$$

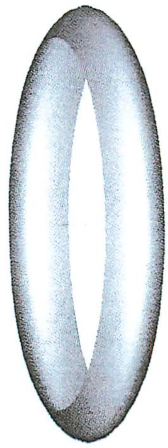
$$B = 4.$$

$$\Rightarrow \boxed{y(x) = e^{-2x}\left(\cos 4x + 4\sin 4x\right)}.$$

432

Part 2: Do these after part 1. **Name:** ______________ **Id:** ______________

15 pts 16. Find the volume of the wedding-band shape obtained by rotating the region between the two curves



$$y = x^2 + 2 \qquad \text{and} \qquad y = 4 - x^2$$

about the horizontal line $y = -3$.

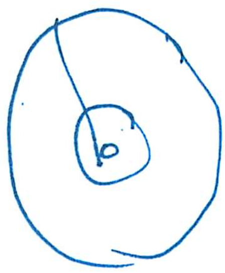
(a) Write an integral which represents the volume.

CROSS AT $x^2+2 = 4-x^2$

$2x^2 = 2 \Rightarrow x = \pm 1.$

AREA (SLICE) IS

$\pi(4-x^2+3)^2 - \pi(x^2+2+3)^2$



$$\boxed{\pi\int_{-1}^{1} (7-x^2)^2 - (x^2+5)^2 \, dx}$$

$49 - 14x^2 + x^4 - [x^4 + 10x^2 + 25] = 24 - 24x^2$

(b) Evaluate the integral in (a).

$\pi\int_{-1}^{1} (24 - 24x^2)\,dx$

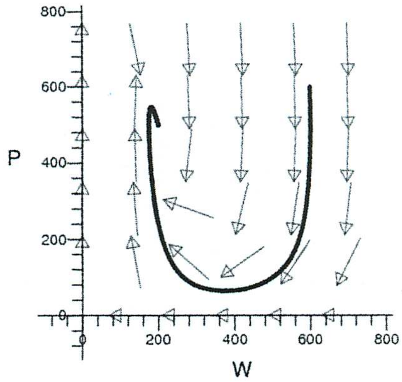
$= \pi\left(24x - 8x^3\right)\Big|_{-1}^{1} = \pi(16+16)$

$= 32\pi.$

439

Part 2: Do these after part 1. **Name:** ______________ **Id:** ______________

$\boxed{\text{15 pts}}$ 17. Two populations, the Pacifists and the Warriors, live near one another. The Pacifists are simple rutabaga farmers: if left to themselves, their population would be well modelled by a logistic growth model. However, the nearby Warriors survive by making regular raids on the Pacifists. The two populations are modelled by the predator-prey system below, where t is in years, $W(t)$ is the population of the Warriors after t years, and $P(t)$ is the population of the Pacifists. The phase portrait for this system is shown at right.



$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{1000}\right) - \frac{PW}{200}$$

$$\frac{dW}{dt} = -\frac{W}{4} + \frac{PW}{2000}$$

(a) Are there any equilibrium solutions? If so, find **all** of them. If not, write "none", and **justify your answer**.

$P' = 2P\left(1 - \frac{P}{1000} - \frac{W}{400}\right)$

$W' = W\left(-\frac{1}{4} + \frac{P}{2000}\right)$

$\Rightarrow W' = 0$ IF $W = 0$ OR $P = 500$

$P' = 0$ IF $P = 0$ OR $1 - \frac{P}{1000} - \frac{W}{400} = 0$ IF

IF $W=0$, $P = 1000/0$

IF $P=500$, $W = 200$.

$\therefore$

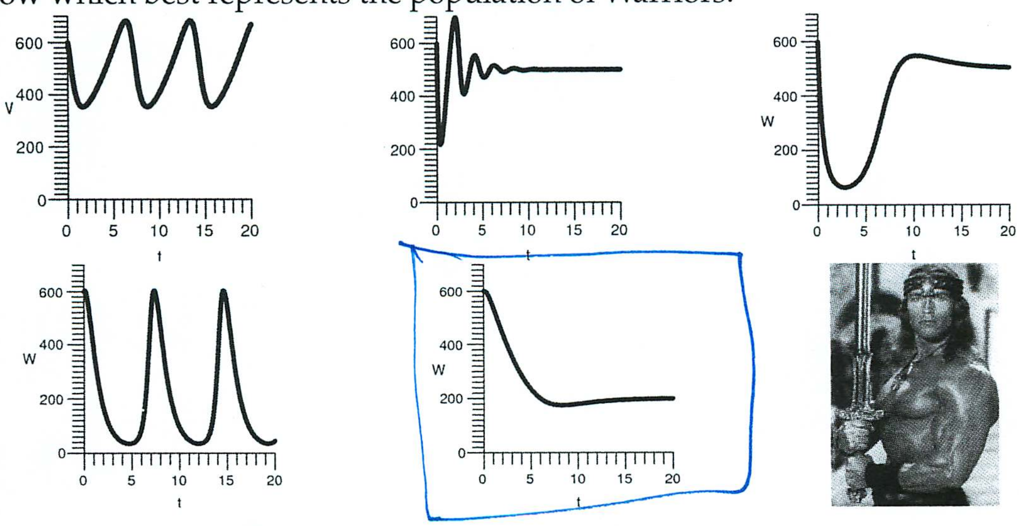
(0,0)

(0, 1000)

(~~200~~, 500)

200

(b) If the populations start out with 600 Pacifists and 600 Warriors, circle the graph below which best represents the population of Warriors.

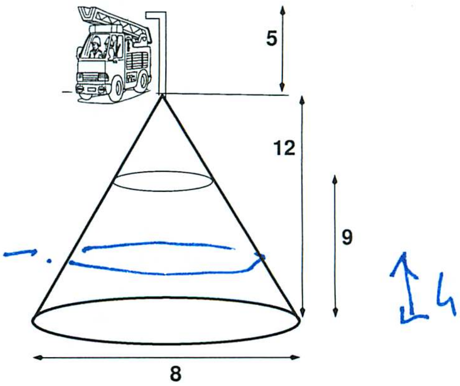


444

Part 2: Do these after part 1. **Name:** ______________ **Id:** ____________

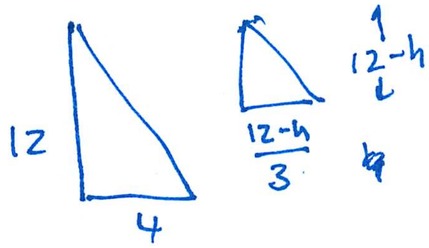
$\boxed{15 \text{ pts}}$ 18. Fireman Fred has an underground tank partially full of Fluorotelomer Fire-Fighting Foam. The tank is conical, with the vertex at the top of the tank. The height of the tank is 12 feet, with a diameter of 8 feet, and is filled to a height of 9 feet. Fred wants to pump the foam out of the tank and into his truck, which fills at a height 5 feet above groundlevel. The foam has a density of 1 pound per cubic foot and has a delightful minty scent.

Write an integral which represents the amount of work required for Fred to pump all of the foam out of the tank and into his truck. (You do not need to calculate the integral).



SLICE @ h.

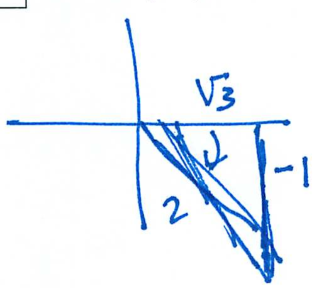
$r = \frac{12-h}{3}$, $V = \pi\left(\frac{12-h}{3}\right)^2$



DISTANCE = $17-h$.

$$\int_0^9 \pi\left(\frac{12-h}{3}\right)^2(17-h)\,dh.$$

$\boxed{15 \text{ pts}}$ 19. Simplify the complex number $(\sqrt{3}-i)^{47}$, writing it in the form $a+bi$ with a and b real.



$\theta = -\frac{\pi}{6}$

$-\frac{47\pi}{6} = \frac{-48\pi+\pi}{6} \approx +\frac{\pi}{6}$

$= 2^{47}\sqrt{3} + 2^{47}i$

Part 2: Do these after part 1. **Name:** ____________ **Id:** ____________

| 15 pts | 20. Let $f(x) = \sqrt{x}$. Find a value c between 4 and 9 so that $f(c)$ is equal to the average of $f(x)$ in $[4, 9]$. You should leave your answer in rough form; it is not necessary to simplify fully.

AVG VAL OF $\sqrt{x}$ ON $[4,9]$ =

$$\frac{1}{5}\int_4^9 x^{1/2}dx = \frac{1}{5}\frac{2}{3}x^{3/2}\Big|_4^9 = \frac{2}{15}(27-8)$$

$$\sqrt{c} = \frac{38}{15}.$$

$$c = \left(\frac{38}{15}\right)^2$$

| 15 pts | 21. State all values of x for which the series $\displaystyle\sum_{n=0}^{\infty} \frac{(2x-3)^n}{n \ln n}$ converges. Don't forget to check the endpoints.

RATIO: $\displaystyle\lim_{n\to\infty}\left|\frac{(2x-3)^{n+1}}{(2x-3)^n}\frac{n\ln n}{(n+1)\ln(n+1)}\right| = 2x-3$

$\Rightarrow -1 < 2x-3 < 1 \Rightarrow 2 < 2x < 4$

$1 < x < 2$

IF $x=1$, $\sum \frac{(-1)^n}{n \ln n}$ CONV, ALT SERIES.

IF $x=2$, $\sum \frac{1}{n \ln n}$ DIVERGES, INTEGRAL TEST,

$\therefore$ $\boxed{1 \le x < 2}$

Name: ________________ Id: ______________

20 pts 22. Compute the following integrals.

(a) $\int \arctan(1/x)\, dx$

$$\frac{x^2}{1+x^2}\left(\frac{-1}{x^2}\right)dx$$

$$u = \arctan\left(\frac{1}{x}\right) \qquad dv = dx$$

$$du = \frac{1}{1+\frac{1}{x^2}}\left(\frac{-1}{x^2}\right)dx \qquad v = x\,dx$$

$$= x\arctan\left(\frac{1}{x}\right) + \int \frac{x\,dx}{1+x^2}$$

$$\boxed{= x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln(1+x^2) + C}$$

(b) $\int \sqrt{16-5x^2}\, dx$ $= \cancel{16}\int \sqrt{1-\left(\frac{\sqrt{5}x}{4}\right)^2}\, dx$

$$\frac{\sqrt{5}x}{4} = \sin\theta, \quad \frac{\sqrt{5}}{4}dx = \cos\theta\, d\theta$$
$$dx = \frac{4}{\sqrt{5}}\cos\theta\, d\theta$$

$$= 16 \cdot \frac{4}{\sqrt{5}}\int \cos^2\theta\, d\theta$$

$$= \frac{\cancel{64}}{\sqrt{5}}\left(\frac{1}{2}\int 1+\cos 2\theta\, d\theta\right)$$

$$= \frac{8\,\cancel{32}}{\sqrt{5}}\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

$$= \frac{\cancel{32}}{\sqrt{5}}\left(\theta + \cancel{\tfrac{1}{2}}\sin\theta\cos\theta + C\right)$$

$$= \frac{\cancel{32}}{\sqrt{5}}\left(\arcsin\left(\frac{\sqrt{5}x}{4}\right) + \frac{\sqrt{5}x}{4}\sqrt{1-\left(\frac{\sqrt{5}x}{4}\right)^2}\right) + C.$$

EWWW.