2006 SAMPLE FINAL SOLUTIONS

EVALUATE THE INTEGRALS.

a) $\int \tan^2 x \sec^4 x \, dx$ USE $1+\tan^2 x = \sec^2 x$

$= \int \tan^2 x \left(1+\tan^2 x\right) \sec^2 x$ LET $u=\tan x$, $du = \sec^2 x\, dx$

$= \int u^2 (1+u^2)\, du = \int u^2 + u^4 \, du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$

$\boxed{= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$

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b) $\int \frac{x^2-1}{x^3+x}\, dx = \int \frac{x^2-1}{x(x^2+1)}\, dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\, dx.$

SO $A(x^2+1) + (Bx+C)x = x^2 - 1$

$(A+B)x^2 = x^2 \Rightarrow A+B = 1$

$Cx = 0x \Rightarrow C = 0$

$A = -1 \Rightarrow A = -1, \ B = 2.$

$\int \frac{x^2-1}{x^3+x}\, dx = \int \frac{-1}{x}\, dx + \int \frac{2x}{x^2+1}\, dx$

$= -\ln|x| + \ln(x^2+1) + C = \boxed{\ln\left|\frac{x^2+1}{x}\right| + C}$

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c) $\int \sqrt{x} \ln x \, dx$ $u = \ln x$, $du = \frac{1}{x}dx$; $dv = \sqrt{x}\, dx$, $v = \frac{2}{3}x^{3/2}$

$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2} dx$

$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{3/2}\right) + C = \boxed{\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C}$

2

d) $\int x^3 \ln x \, dx$

$u = \ln x \qquad dv = x^3$

$du = \frac{1}{x} dx \qquad v = \frac{1}{4}x^4$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx = \boxed{\frac{1}{4}x^4 \ln x - \frac{x^4}{16} + C}$$

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e) $\int \tan(10x)\, dx$

~~u = 10x, du~~ ⇒

$$= \int \frac{\sin(10x)}{\cos(10x)} dx$$

$u = \cos 10x$

$du = -10 \sin 10x \, du$

$$= -\frac{1}{10}\int \frac{du}{u} = -\frac{1}{10}\ln u + C = \boxed{-\frac{1}{10}\ln|\cos(10x)| + C}$$

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f) $\int x \sin^2 x \cos x \, dx$

$u = x \qquad dv = \sin^2 x \cos x \, dx$

$du = dx \qquad v = \frac{\sin^3 x}{3}$

$$= \frac{1}{3}x \sin^3 x - \frac{1}{3}\int \sin^3 x \, dx$$

USE $\sin^2 x + \cos^2 x = 1$

$$= \frac{1}{3}x \sin^3 x - \frac{1}{3}\int (1 - \cos^2 x) \sin x \, dx$$

$$= \frac{1}{3}x \sin^3 x + \frac{1}{3}\int (1 - u^2)\, du$$

LET $u = \cos x$

$du = -\sin x \, dx$

$$= \boxed{\frac{1}{3}x \sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C}$$

SOL'NS 3

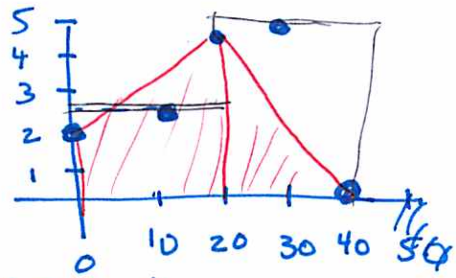
2

f GIVEN BY TABLE

x	0	10	20	30	40
$f(x)$	2	2.5	4.5	5	0

a) USE TRAPEZOID RULE w/ 2 DIVISIONS TO APPROXIMATE (RED)

$$\int_0^{40} f(x)\,dx.$$



WIDTH = 20

$$T_2 = (20)\left(\tfrac{1}{2}\right)(2 + 4.5 + 4.5 + 0)$$

$$= (10)(11) = \boxed{110}$$

b) MIDPOINT w/ 2 DIVISIONS (BLACK)

$$M_2 = 20 \cdot 2.5 + 20 \cdot 5$$

$$= 50 + 100 = \boxed{150}$$

3

ARC LENGTH OF GRAPH OF $y = x^{3/2}$, $100 < x < 400$

a) $$\int_{100}^{400} \sqrt{1 + \left(\tfrac{3}{2} x^{1/2}\right)^2}\,dx = \int_{100}^{400} \sqrt{1 + \tfrac{9}{4}x}\,dx$$

b) $$\int_{1/9}^{1/4} \sqrt{1 + \tfrac{9}{4}x}\,dx =$$

LET $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}\,dx$

$x = \frac{1}{9} \Rightarrow u = \frac{5}{4}$

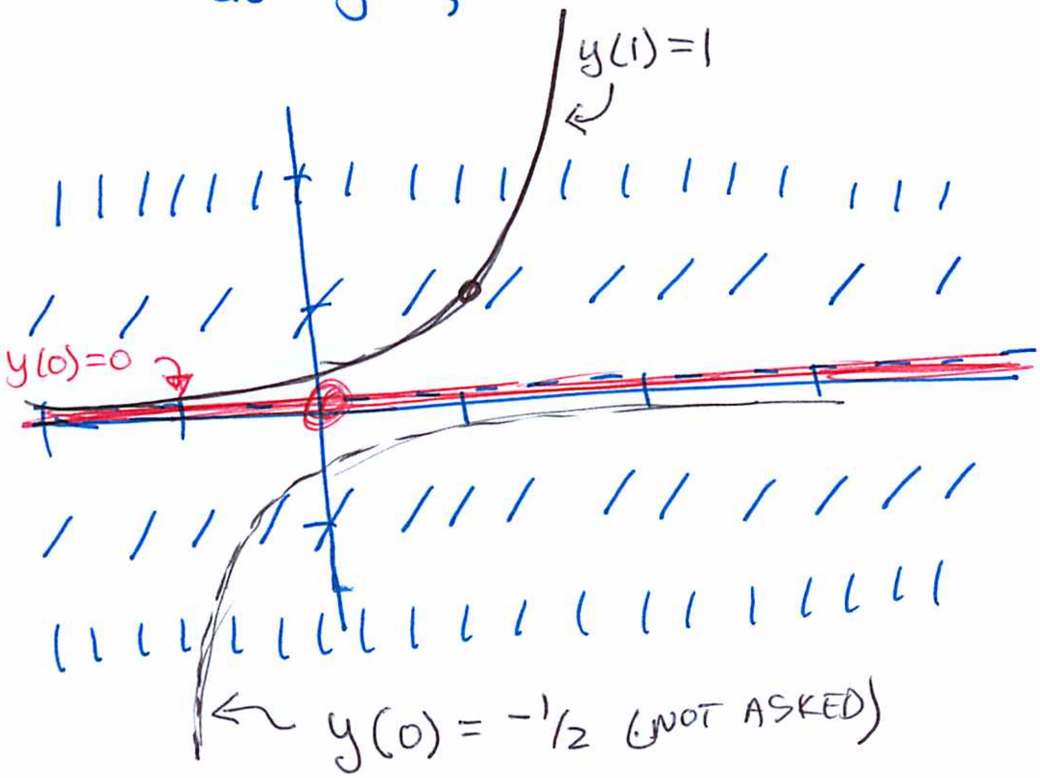
$x = \frac{1}{4} \Rightarrow u = \frac{25}{16}$

$$= \frac{4}{9} \int_{5/4}^{25/16} u^{1/2}\,du$$

$$= \frac{4}{9} \cdot \frac{2}{3}\, u^{3/2} \Big|_{5/4}^{25/16} = \boxed{\frac{8}{27}\left(\left(\frac{5}{4}\right)^3 - \left(\frac{5}{4}\right)^{3/2}\right)} = \frac{1}{27}\left(125 - 20\sqrt{5}\right)$$

#4) DRAW THE DIRECTION FIELD FOR

$\frac{dy}{dt} = y^2$, & SKETCH SOL'NS w $y(1) = 1$ & $y(0) = 0$



$y = 2 \Rightarrow y' = 4$

$y = 1 \Rightarrow y' = 1$

$y = 0 \Rightarrow y' = 0$

$y = -1 \Rightarrow y' = +1$

$y = -2 \Rightarrow y' = +4$

NOT ASKED: GEN'L SOLUTION:

$$\frac{dy}{dx} = y^2$$

$$\int \frac{dy}{y^2} = \int dx$$

$$-\frac{1}{y} = x + C$$

$$y = \frac{-1}{x+C} \quad (\text{IF } y \neq 0)$$

IF $y(0) = 0$, $y' = 0$, SO $y(x) = 0$ FOR ALL X

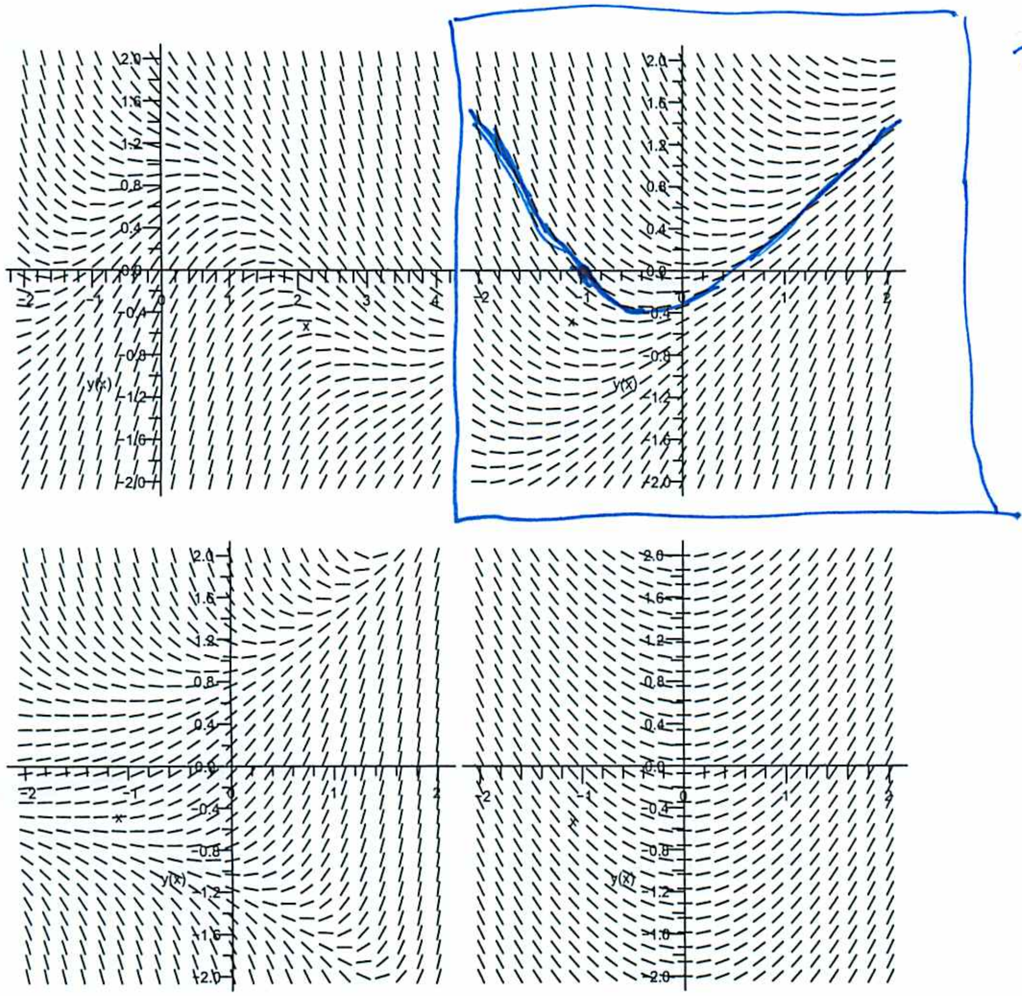
IF $y(1) = 1$, $1 = \frac{-1}{1+C} \Rightarrow C = -2$, SO $y(x) = \frac{-1}{x-2}$

IF $y(0) = -\frac{1}{2}$, $-\frac{1}{2} = \frac{-1}{C} \Rightarrow C = 2$, SO $y(x) = \frac{-1}{x+2}$

#5. WHICH DIAGRAM CORRESPONDS

a) TO $y' = x - y$?

THIS ONE.

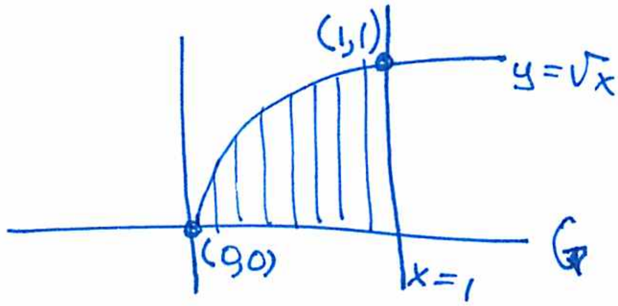


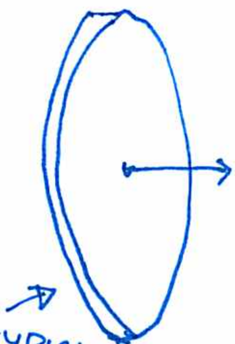
THIS CAN BE SEEN SINCE, FOR EXAMPLE ALONG THE LINE $y = x$ WE HAVE SLOPE 0

b) SKETCH THE SOLUTION WITH $y(-1) = 0$ ON THE DIAGRAM FOR (a).

6a SKETCH THE REGION & FIND THE VOLUME OF THE SOLID FORMED BY REVOLVING IT AROUND THE x-AXIS.

a) $y=\sqrt{x}$, $y=0$, $x=1$





VOLUME IS

$$\int \begin{pmatrix}\text{AREA}\\ \text{OF}\\ \text{SLICE}\end{pmatrix} dx$$

$$= \int_0^1 \pi(\sqrt{x})^2 dx$$

$$= \pi \int_0^1 x\,dx = \frac{\pi}{2}x^2 \Big|_0^1 = \boxed{\frac{\pi}{2}}$$

A TYPICAL SLICE AT A GIVEN x IS A DISK OF RADIUS $\sqrt{x}$

AREA $= \pi(\sqrt{x})^2$

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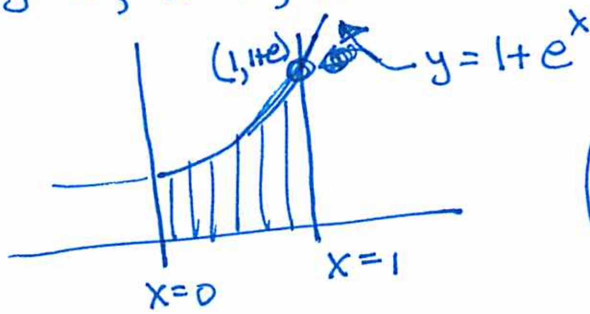
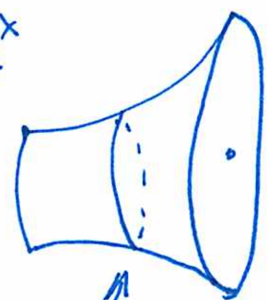
b) $y = 1+e^x$, $y=0$, $x=0$, $x=1$





VOLUME =

$$\int_0^1 \pi(1+e^x)^2 dx$$

$$= \pi\int_0^1 (1+2e^x+e^{2x})\,dx$$

$$= \pi\left(x+2e^x+\tfrac{1}{2}e^{2x}\right)\Big|_0^1$$

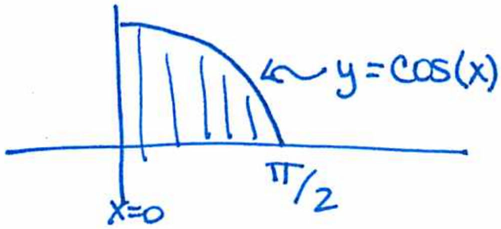
$$= \pi\left[\left(1+2e+\tfrac{1}{2}e^2\right)-\left(0+2+\tfrac{1}{2}\right)\right]$$

$$= \pi\left(2e+\tfrac{1}{2}e^2-\tfrac{3}{2}\right)$$

TYPICAL SLICE IS DISK OF RADIUS $1+e^x$,

AREA $= \pi(1+e^x)^2$

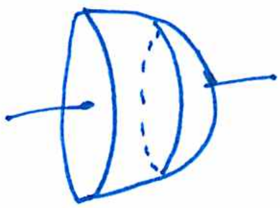
7a) SKETCH THE ~~AREA~~ REGION BOUNDED BY $y = \cos(x)$, THE X-AXIS, $X=0$ AND $x = \pi/2$



b) FIND THE AREA.

$$\int_0^{\pi/2} \cos(x)\,dx = \sin(x)\Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = \boxed{1}$$

c) COMPUTE THE VOLUME OF THE SOLID OBTAINED BY REVOLVING THIS REGION AROUND THE X-AXIS.



A TYPICAL SLICE IS A DISK OF RADIUS $\cos x$.
AREA IS $\pi \cos^2 x$

VOLUME IS

$$\int_0^{\pi/2} \pi \cos^2 x\,dx = \pi \int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2x)\,dx$$

$$= \frac{\pi}{2}\left(x + \tfrac{1}{2}\sin 2x\right)\Big|_0^{\pi/2} = \frac{\pi}{2}\cdot\frac{\pi}{2} = \boxed{\frac{\pi^2}{4}}$$

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8) SPOZE $y' = y^2/x$

a) GIVE THE GENERAL SOLUTION: $\dfrac{dy}{dx} = \dfrac{y^2}{x}$

SO $\displaystyle\int \frac{dy}{y^2} = \int \frac{dx}{x}$

$$-\frac{1}{y} = \ln x + C$$

$$\boxed{y = \frac{1}{C - \ln x}}$$

b), IF $y(1)=0$, FIND $y(x)$.

SINCE $y(1)=0$, $y' = \frac{0}{1} = 0$.

THUS $y(x)$ IS A CONSTAND SO

$$\boxed{y(x) = 0}$$

9) FOR $y' = y - x^2$,

WHICH IS A SOLUTION:

i) $y = \cos(cx)$: $\boxed{\text{NO}}$; $y' = c\sin(cx)$,
WHICH IS NOT EQUAL TO
$\cos(cx) - x^2$.

ii) $y = 2 + 2x + Cx^2 + e^x$

$y' = 2 + 2Cx^2 + e^x$ $\boxed{\text{NO}}$ SINCE THIS IS NOT $=$
$(2 + 2x + Cx^2 + e^x) - x^2$

iii) $y = 2 + 2x + x^2 + Ce^x$ $\boxed{\text{YES}}$, SINCE

$y' = 2 + 2x + Ce^x$ $\quad y' = (2 + 2x + x^2 + Ce^x) - x^2$

9b) IF $y(0) = 3$, WE HAVE $3 = 2 + 0 + 0 + Ce^0$, SO $C = 1$.
THUS, SOL'N IS
$y(x) = 2 + 2x + x^2 + e^x$

10) LET $\{a_n\}_{n=1}^{\infty}$ BE $\left\{\frac{\cos(n)}{2^n}\right\}_{n=1}^{\infty}$

a) $a_1 = \frac{\cos(1)}{2}$, $a_3 = \frac{\cos(3)}{2^3}$, $a_{10} = \frac{\cos(10)}{2^{10}}$

b) $\lim_{n\to\infty} \frac{\cos(n)}{2^n} \to 0$, SINCE $-\frac{1}{2^n} \leq \frac{\cos(n)}{2^n} \leq \frac{1}{2^n}$

AND $\lim_{n\to\infty} \frac{-1}{2^n} = 0 = \lim_{n\to\infty} \frac{1}{2^n}$.

c) DOES $\sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$ CONVERGE.

11 WRITE 0.555... AS A GEOMETRIC SERIES AND WRITE AS A FRACTION.

$$0.555\cdots = .5\left(1+\frac{1}{10}+\frac{1}{100}+\cdots\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = \frac{1/2}{1-1/10}$$

$$= \frac{1}{2}\frac{1}{9/10} = \frac{1}{2}\cdot\frac{10}{9} = \boxed{\frac{5}{9}}$$

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12 WRITE 0.171717... AS A FRACTION

$$0.171717\cdots = \frac{17}{100}+\frac{17}{(100)^2}+\frac{17}{(100)^3}+\cdots = \frac{17}{100}\left(1+\frac{1}{100}+\frac{1}{(100)^2}+\cdots\right)$$

$$= \frac{17}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n = \frac{17}{100}\,\frac{1}{99/100} = \boxed{\frac{17}{99}}$$

DO THESE CONVERGE?

13 a) $$\sum_{n=1}^{\infty}\frac{1+n^2 2^{-n}}{n^2} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{2^n}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{2^n}$$

$\boxed{\text{CONVERGES}}$, SINCE $\sum_1 \frac{1}{n^2}$ CONVERGES (P-SERIES, P=2)

$\sum_1 \frac{1}{2^n}$ CONVERGES (RATIO TEST, SEE PROBLEM 10)

b) $\sum_{n=1}^{\infty}\frac{2n+1}{n^2+n}$ $\boxed{\text{DIVERGES}}$ BY LIMIT COMPARISON W/TH $\frac{1}{n}$

$$\lim_{n\to\infty}\frac{2n+1/(n^2+n)}{1/n} = \lim_{n\to\infty}\frac{2n^2+n}{n^2+n} = 2 \neq 0,$$

SO $\sum_1 \frac{1}{n}$ AND $\sum_1 \frac{2n+1}{n^2+n}$ DIVERGE TOGETHER

(OR JUST COMPARISON: $\frac{2n+1}{n^2+n} > \frac{2n}{n^2+n} > \frac{2}{n+1} > \frac{2}{2n} = \frac{1}{n}$)

13c) $\sum \frac{\cos(n)}{n^2}$ $\boxed{\text{CONVERGES}}$

SINCE $\left|\frac{\cos(n)}{n^2}\right| \le \frac{1}{n^2}$, AND $\sum \frac{1}{n^2}$ CONVERGES $\begin{pmatrix}\text{P-SERIES}\\ P=2\end{pmatrix}$

13d) $\sum \frac{2^n}{n!}$ $\boxed{\text{CONVERGES}}$

RATIO TEST: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$

SINCE $0<1$, THE SERIES CONVERGES.

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14 a) $f(x) = \frac{1}{1-x}$. WRITE A SERIES FOR f FOR $|x-3|<1$ (IE, CENTERED AT 3)

FIND TAYLOR SERIES AT $a=3$.

$f(x) = (1-x)^{-1}$ $\quad f(3) = \frac{-1}{2}$

$f'(x) = (1-x)^{-2}$ $\quad f'(3) = \frac{1}{2^2}$

$f''(x) = 2(1-x)^{-3}$ $\quad \frac{f''(3)}{2} = \frac{-1}{2^3}$

$f^{(3)}(x) = 3!(1-x)^{-4}$ $\quad \frac{f'''(3)}{3!} = \frac{1}{2^4}$

$\vdots$

$f^{(n)}(x) = n!(1-x)^{-n-1}$ $\quad \frac{f^{(n)}(3)}{n!} = \frac{(-1)^{n+1}}{2^{n+1}}$

$$\therefore \boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}}(x-3)^n}$$

b) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} n(x-3)^{n-1}$

SO

$$\boxed{(x-3)f'(x) = \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{2^{n+1}}(x-3)^n}$$

15 a) WRITE $\frac{1}{2+x^2}$ AS A GEOMETRIC SERIES

$$\frac{1}{2+x^2} = \frac{1}{2}\left(\frac{1}{1+\frac{x^2}{2}}\right) = \frac{1}{2}\left(\frac{1}{1-\left(\frac{-x^2}{2}\right)}\right) = \boxed{\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x^2}{2}\right)^n}$$

$$= \frac{1}{2}\left(1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{2^3} + \frac{x^8}{2^4} - \cdots \right)$$

b) WHAT IS $f^{(20)}(0)$?

SINCE THE ~~20th~~ COEFFICIENT OF x^{20} IN THE MACLARIN SERIES IS $\frac{f^{(20)}(0)}{20!}x^{20}$.

THIS TERM IS $+\frac{x^{20}}{2^{11}}$, SO $\boxed{f^{(20)}(0) = \frac{20!}{2^{11}}}$

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16 FOR WHICH VALUES OF x DO THE FOLLOWING CONVERGE?

a) $\sum_{n=1}^{\infty} \frac{n}{e^n}(x-3)^n$.   RATIO TEST: $\lim_{n\to\infty}\left|\frac{n+1}{e^{n+1}}(x-3)^{n+1}\cdot\frac{e^n}{n}\frac{1}{(x-3)^n}\right|$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\frac{(x-3)}{e}\right| = \frac{|x-3|}{e}$$

THIS WILL BE <1 WHEN $|x-3| < e$, IE, $3-e < x < e+3$

IF $x = e+3$, SERIES BECOMES

$$\sum_{n=1}^{\infty}\frac{n\,e^n}{e^n} = \sum n$$

DIVERGES.

IF $x = 3-e$

$$\sum \frac{n(-e)^n}{e^n} = \sum (-1)^n n$$

$\lim \neq 0$ ← DIVERGES

$\boxed{\text{CONV FOR } x \in (3-e, 3+e)}$

16b) $\sum_{n=1}^{\infty} \frac{\ln n}{\ln(n+1)} x^n$

RATIO: $\lim_{n\to\infty} \left| \frac{\ln(n+1)}{\ln(n+2)} x^{n+1} \cdot \frac{\ln(n+1)}{\ln(n)\; x^n} \right| = \lim_{n\to\infty} \left| \frac{\ln(n+1)}{\ln(n+2)} \frac{\ln(n+1)}{\ln(n)} x \right|$

$= |x|$, SO OK FOR $-1 < x < 1$

~~CONV~~

IF $x = 1$, SERIES IS

$\sum \frac{\ln n}{\ln(n+1)}$ DIVERGES, LIMIT $a_n \not\to 0$.

IF $x = -1$ SERIES IS

$\sum (-1)^n \frac{\ln n}{\ln(n+1)}$ DIVERGES LIMIT $\not\to 0$.

16c) $\sum_{n=1}^{\infty} \frac{3^n}{n^3} x^n$

RATIO: $\lim_{n\to\infty} \left| \frac{3^{n+1} \; x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n x^n} \right| = \lim_{n\to\infty} \left| \frac{n^3}{(n+1)^3} 3x \right|$

$= |3x|$.

LESS THAN 1 FOR

$-\frac{1}{3} < x < 1/3$

WHEN $x = \frac{1}{3}$,

$\sum_{n=1}^{\infty} \frac{3^n}{n^3} \left(\frac{1}{3}\right)^n = \sum \frac{1}{n^3}$

CONVERGES

(P SERIES, P=3

WHEN $x = -\frac{1}{3}$

$\sum \frac{(-1)^n}{n^3}$

CONVERGES ALT SERIES

$\boxed{x \in \left[\frac{-1}{3}, \frac{1}{3}\right]}$

17 FIND THE EXACT VALUE OF

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

WE SHOULD RECOGNIZE THIS AS $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ WITH $x=2$. SINCE $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, THE SUM IS

$$\boxed{e^2}$$

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18 EVALUATE $\int \frac{\sin x}{x} dx$ AS A SERIES.

RECALL $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

SO $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

SO $\int \frac{\sin x}{dx} = C + x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \dots = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$

19 a) FIND ALL REAL-VALUED FUNCTIONS $y(x)$ SATISFYING $y'' + y' + 2y = 0$.

WRITE CHAR. POLY $\lambda^2 + \lambda + 2 = 0$

USING THE QUADRATIC FORMULA, WE GET $\lambda = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1}{2} \pm \frac{\sqrt{7}}{2} i$

THIS MEANS $y(x) = A_1 e^{\frac{-1+\sqrt{7}i}{2} x} + B_1 e^{\frac{-1-\sqrt{7}i}{2}}$

IS THE ~~seen~~ GENERAL SOL'N, BUT WE WANT REAL ONES, SO

$$\boxed{y(x) = e^{-\frac{1}{2}x}\left(A \cos\left(\frac{\sqrt{7}}{2}x\right) + B \sin\left(\frac{\sqrt{7}}{2}x\right)\right)}$$

b) FIND SOL'NS THRU $(0, 1)$. HOW MANY?

SINCE $y(0) = 1$, WE HAVE

$1 = A$. SO SUCH SOLUTIONS HAVE THE FORM

$$\boxed{y = Ae^{-1/2x} \cos\left(\frac{\sqrt{7}}{2}x\right)}$$

AND THERE ARE ∞-MANY OF THEM.

c) WHAT IF $y'(0) = -1$?

WE HAVE $y'(x) = \frac{-1}{2} e^{-1/2x}\left(A \cos\left(\frac{\sqrt{7}}{2}x\right) + B \sin\left(\frac{\sqrt{7}}{2}x\right)\right)$

SO $+ e^{-1/2x}\left(-A \sin\left(\frac{\sqrt{7}}{2}x\right) \frac{\sqrt{7}}{2} + B\frac{\sqrt{7}}{2} \cos\left(\frac{\sqrt{7}}{2}x\right)\right)$

$-1 = y'(0) = \frac{-1}{2}(A) + \frac{\sqrt{7}}{2}B$. THUS $A = 2 + B\sqrt{7}$.

SOL'N IS $y(x) = e^{-x/2}\left((2+B\sqrt{7}) \cos\left(\frac{\sqrt{7}}{2}x\right) + B\sin(\ldots\right)$