2. $\boxed{\text{12 points}}$ Find the Maclaurin series of each of the given functions. I suggest you use a familiar power series as your starting point.

(a) e^{3x^2}

SINCE $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

NOW SUBSTITUTE $3x^2$ IN FOR x IN THE ABOVE.

$$e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!} = 1 + 3x^2 + \frac{9}{2}x^4 + \frac{27}{6}x^6 + \cdots$$

(b) $\dfrac{1}{1+4x}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$\frac{1}{1+4x} = \frac{1}{1-(-4x)}$ SO WE SUBSTITUTE $-4x$ FOR x ABOVE TO GET

$$\sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^n = 1 - 4x + 16x^2 - 64x^3 + \cdots$$

(c) $\sin 2x$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

SO SUBSTITUTING IN $2x$, WE GET

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$$

3. $\boxed{\text{12 points}}$ For each of the series below, decide if it converges or diverges. **You must justify your answer to receive full credit.**

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n+3^n} = \frac{1}{4} + \frac{2}{17} + \frac{1}{19} + \frac{8}{353} + \ldots$

CONVERGES. WE CAN COMPARE IT TO THE CONVERGENT GEOMETRIC SERIES $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n$, SINCE $\displaystyle\frac{2^n}{5^n+3^n} < \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$

(OR, IF YOU PREFER LIMIT COMPARISON,

$$\lim_{n\to\infty} \frac{2^n/(5^n+3^n)}{2^n/5^n} = \lim_{n\to\infty} \frac{5^n}{5^n+3^n} = 1 \text{ SO}$$

BOTH CONVERGE)

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n} = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \ldots$

$\dfrac{1}{x\ln x}$ IS DECREASING FOR $x \geq 2$, POSITIVE AND CONTINUOUS, SO WE CAN USE THE INTEGRAL TEST.

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{M\to\infty}\int_2^M \frac{dx}{x\ln x} \quad \boxed{\begin{aligned}&\text{LET } u=\ln x\\ &du = \frac{dx}{x}\end{aligned}} \quad = \lim_{M\to\infty}\int_{\ln 2}^{\ln M}\frac{du}{u}$$

$$= \lim_{M\to\infty} \ln u \Big|_{\ln 2}^{\ln M} = \lim_{M\to\infty}\left(\ln(\ln M) - \ln(\ln 2)\right) = +\infty$$

SO THE SERIES DIVERGES

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!} = 1 + 5 + \frac{500}{3} + \frac{1250}{3} + \ldots$

USE THE RATIO TEST:

$$\lim_{n\to\infty}\left|\frac{10^{n+1}}{(n+1)!}\cdot\frac{n!}{10^n}\right| = \lim_{n\to\infty}\left|\frac{10\cdot 10^n}{10^n}\cdot\frac{n!}{n\cdot n!}\right| = \lim_{n\to\infty}\left|\frac{10}{n}\right| = 0$$

SINCE THE RATIO IS 0, THE SERIES CONVERGES.

OR RECOGNIZE THIS AS THE MACLAURIN SERIES FOR e^{10x}.

Name: ____________________ Id: ____________________

4. $\boxed{\text{16 points}}$ Let $f(x) = x^{1/3}$.

(a) Find the Taylor polynomial of degree 2 centered at $a = 8$ for $f(x)$.

$f(x) = x^{1/3} \quad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3} \quad f'(8) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3} \quad f''(8) = -\frac{2}{9}\cdot\frac{1}{32} = -\frac{1}{9\cdot 16}$

SO THE 2ND TAYLOR POLYNOMIAL IS

$$2 + \frac{1}{12}(x-8) - \frac{1}{18\cdot 16}(x-8)^2$$

QUESTIONS LIKE THIS PART WILL NOT BE ON THE MIDTERM

(b) If the result of the previous part is used to estimate $9^{1/3}$, what is the error (according to Taylor's theorem)?

TAYLOR'S THEOREM SAYS THE ERROR WILL BE NO MORE THAN $\left|\frac{f'''(c)}{3!}\right|$ FOR SOME $8 \leq c \leq 9$

SINCE $f'''(x) = \frac{10}{27}x^{-8/3}$ IS DECREASING, THE MAX WILL OCCUR AT $x=8$. SO THE ERROR IS NO MORE THAN

$\frac{10}{27\cdot 2^8\cdot 3!}$, WHICH IS PRETTY GOOD.