Question 1

Does an increasing sequence always converge if it is bounded below?

If yes, give a brief explanation, and illustrate with a plot of a sequence.

If no, give an example of a bounded below, increasing sequence that diverges, and illustrate with a plot.

(An approximate sketch suffices, the plot does not need to be up to scale.)

Question 2

Consider a series $\sum_{n=1}^{\infty} b_n$.

Its sequence of its partial sums is $\{S_n\}$, such that $S_n = b_1 + b_2 + b_3 + \cdots + b_n$.

(a) Suppose the sequence of the partial sums is given by $S_n = \dfrac{3n^2 - 2n}{1 - 2n^2}$.

Does the sequence of the partial sums $\{S_n\}$ converge? If converges, compute $\lim_{n\to\infty} S_n$. Please show all work.

(b) Using your result in part (a), determine whether the series $\sum_{n=1}^{\infty} b_n$ converges. If converges, find its sum.

(c) Using the values of the partial sums S_1 and S_2, find the terms b_1 and b_2.

(d) Does the sequence $\{b_n\}$ converge or diverge? If converges, find $\lim_{n\to\infty} b_n$.

Hint: think about the Divergence Test.

In this question, you work with two sequences:

- $\{S_n\}$ *of the partial sums;*
- $\{b_n\}$ *of the terms of the series.*

Please remember that they are distinct.

Show all work and explain your reasoning in all parts of the question.

Question 3

The parts of the question may be solved independently in any order, but it's fastest to solve them in the order they are given.

For all parts of the question, show all work and explain your reasoning. Unexplained answers will receive little credit.

This question is about a series that depends on a parameter c. The parameter can take any values.

Consider the series

$$\sum_{n=0}^{\infty} \frac{(7c)^{n+1}}{2^{2n+1}}.$$

(a) Describe *all* values of the parameter c for which the series converges. Plot these values on the number line (you should get an interval).

(b) For each of the following values of c, determine if the series converges or diverges.

$$c = -\frac{3}{2} \qquad c = \frac{2}{3} \qquad c = 0 \qquad c = \frac{1}{2} \qquad c = \frac{4}{7} \qquad c = -\frac{3}{20}$$

(c) Pick any value of c, different from those on the list above, for which your series converges. Compute the sum of the series for your chosen c.

Question 4

Hermione wants to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges or diverges.

She uses the Integral Test but makes some mistakes in her calculations and arguments.

For each of the steps below, decide whether her work is correct. For each step that contains mistakes, please explain why Hermione's calculation or reasoning is incorrect, and provide a correct and complete solution for that step. Make sure to label the steps with letters A-F, as below. Include a clear statement: "Steps X, Y, and Z are incorrect", and then elaborate on each step separately. If making a picture, a rough sketch suffices, but be sure to label all the relevant points. Correct steps require no explanation.

A. She takes the corresponding function $f(x) = \frac{x}{(x^2+1)^2}$. Looking at the graph, she decides that f satisfies all conditions required by the Integral Test: this function is continuous, positive, and decreasing.
This step requires no further elaboration.

B. She then computes the antiderivative of f, using the substitution $u = x^2+1$, $du = xdx$,

$$\int \frac{xdx}{(x^2+1)^2} = \int \frac{du}{u^2} = -\frac{1}{2u} + C = -\frac{1}{2(x^2+1)} + C$$

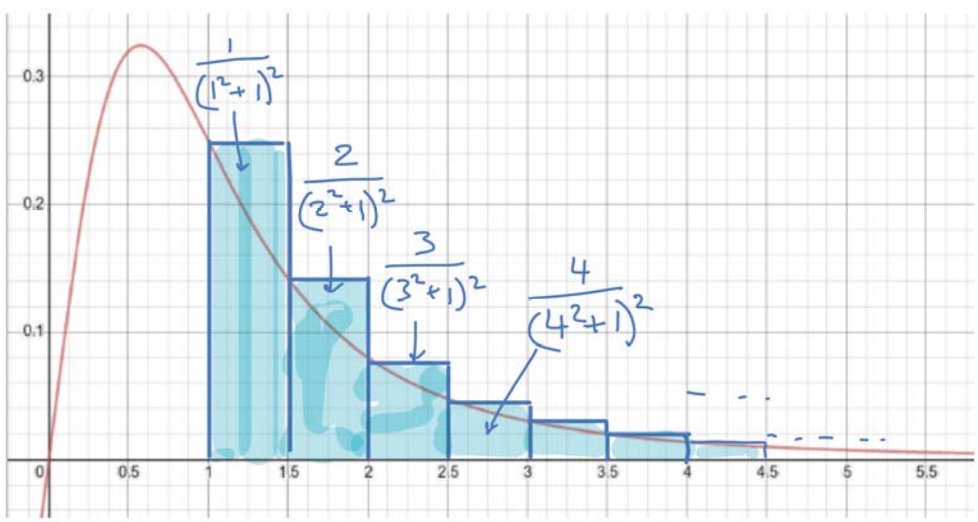
C. and evaluates the improper integral:

$$\int_1^{+\infty} \frac{xdx}{(x^2+1)^2} = -\frac{1}{2(x^2+1)} \Big|_1^{+\infty} = -\frac{1}{2(\infty^2+1)} - \left(-\frac{1}{2(1^2+1)}\right) = 0 + \frac{1}{4} = \frac{1}{4}$$

so she decides that the improper integral converges.

D. She then applies the Integral Test: since the corresponding integral converges, the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ also converges.

E. To explain how the comparison of the integral and the series works in this example, she represents graphically the terms of the series as the areas of the collection of rectangles, as shown:



F. Hermione explains why the convergence of the integral imply the convergence of series:
"The area of the rectangles represent the series as a Riemann sum for the integral. We know that the integral converges, so the area under the graph is finite. The Riemann sums give an approximation for the area under the graph, therefore, the Riemann sum representing the series must be also finite. Therefore, the series converges."