1. *(15 pts)*

(a) Determine whether the following sequence converges. If it converges, find the limit.

$$a_n = \frac{n^3+4}{n^3-2n^2+4}$$

SINCE $\lim_{n\to\infty}\left(\frac{n^3+4}{n^3-2n^2+4}\right) = 1$

THE SEQUENCE CONVERGES TO 1

(b) Determine whether the following sequence converges. If it converges, find the limit.

$$a_n = \frac{n^2+3}{e^n}$$

WE CAN DO THIS A COUPLE OF WAYS.

$\lim_{n\to\infty} \frac{n^2+3}{e^n}$ IS OF THE FORM $\frac{\infty}{\infty}$, SO WE COULD USE L'HOPITAL'S RULE TWICE TO GET $\lim_{n\to\infty} \frac{2}{e^n} = 0$.

OR:

OBSERVE ~~THAT~~ THE TERMS ARE DECREASING AND POSITIVE (SO BOUNDED BELOW BY 0) AND HENCE CONVERGE.

TO SEE THAT THEY CONVERGE TO ZERO STILL NEEDS SOME EFFORT, FOR EXAMPLE L'HOPITALS OR SHOW THAT $\frac{n^2+3}{e^n}$ GETS AS SMALL AS YOU LIKE FOR n BIG.

BUT THAT IS WORK AND I DON'T WANT TO.

(c) Consider the sequence given by $a_1 = \sqrt{2}$, $a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}}$, $a_3 = \sqrt{2a_2}$, ..., $a_{n+1} = \sqrt{2a_n}$. It is known that this sequence is monotonically increasing and bounded above. Is the sequence $\{a_n\}_n$ convergent? Justify your answer. If it converges, find the limit.

SINCE IT IS MONOTONE INCREASING AND BOUNDED, IT MUST CONVERGE. ~~TO~~ SOME LIMIT L. TO FIND L, NOTE THAT $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$.

BUT $a_{n+1} = \sqrt{2a_n}$, SO

$$L = \lim_{n\to\infty} a_{n+1} = \sqrt{2 \lim_{n\to\infty} a_n} = \sqrt{2L}$$

IE $L = \sqrt{2L}$. SQUARING BOTH SIDES GIVES $L^2 = 2L$, IE L IS A SOLUTION TO $L^2 - 2L = 0$

$$L(L-2) = 0$$

SO L IS 0 OR 2.

BUT SINCE $a_n > 0$ FOR ALL n, $L = 2$.

$\boxed{\text{THE LIMIT IS 2}}$

2. *(15 pts)*

(a) Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1} = \frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{25} + \cdots$$

THIS IS AN ALTERNATING SERIES.

NOTE THAT THE TERMS ARE DECREASING IN ABS. VALUE

AND $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$.

SO THIS CONVERGES BY THE ALT. SERIES TEST.

(b) Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^3+1}{n^4+2n+3}$$

DIVERGES BY LIMIT COMPARISON WITH $\sum_1^\infty \frac{1}{n}$.

$$\lim_{n\to\infty}\left(\frac{\frac{n^3+1}{n^4+2n+3}}{1/n}\right) = \lim_{n\to\infty} \frac{n^4+n}{n^4+2n+3} = 1$$

SO $\sum_1^\infty \frac{1}{n}$ & $\sum_1^\infty \frac{n^3+1}{n^4+2n+3}$ BOTH DIVERGE ($\sum_1^\infty \frac{1}{n}$ DIV. SINCE HARMONIC SERIES)

YOU COULD USE DIRECT COMPARISON, BUT NOT WITH $\sum_1^\infty \frac{1}{n}$. HOWEVER, DIRECT COMPARISON WITH $\frac{1}{2}\sum_1^\infty \frac{1}{n} = \sum_1^\infty \frac{1}{2n}$ WORKS.

$$\frac{1}{2n} < \frac{n^3+1}{n^4+2n+3}, \text{ SINCE } n^4+2n+3 < 2n^4+2n \text{ FOR } n>2.$$

SINCE $\sum_1^\infty \frac{1}{2n}$ DIVERGES SO DOES $\sum_1^\infty \frac{n^3+1}{n^4+2n+3}$.

(c) Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

HERE, WE HAVE TO USE THE INTEGRAL TEST.

NOTE THAT $\frac{1}{x(\ln x)^2}$ IS POSITIVE, DECREASING AND CONTINUOUS

FOR $x > 2$.

LET $u = \ln x$
$du = \frac{dx}{x}$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2} = \lim_{m\to\infty}\left(-\frac{1}{u} \Big|_{\ln 2}^{m} \right)$$

$$= \lim_{m\to\infty}\left(-\frac{1}{m} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$$

SO THE INTEGRAL CONVERGES.

HENCE $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ CONVERGES.

3. *(20pts)*

(a) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

(Hint: factor the denominator and write it as a telescopic sum!)

USING PARTIAL FRACTIONS. SEE BELOW.

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

$$\text{SO } \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \left(\frac{1}{2} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{10}} - \cancel{\frac{1}{14}}\right) + \cancel{\ldots} \ldots$$

$$= \frac{1}{2}$$

PARTIAL FRACTIONS

$$\frac{1}{4n^2-1} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

ie $1 = (2n+1)A + (2n-1)B$

LET $n = \frac{1}{2}$: $1 = 2A \Rightarrow A = \frac{1}{2}$

LET $n = -\frac{1}{2}$: $1 = -2B \Rightarrow B = -\frac{1}{2}$

$$\text{SO } \frac{1}{4n^2-1} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

(b) Write the number $2.1\overline{79} = 2.1797979\ldots$ as a fraction.

THINK OF $.0797979\ldots$ AS A GEOMETRIC SERIES WITH RATIO $\frac{1}{100}$.

$$.0797979\ldots = .079 + \frac{.079}{100} + \frac{.079}{100^2} + \cdots$$

$$= \frac{.079}{1 - 1/100} = .079\left(\frac{100}{99}\right) = \frac{79}{990}$$

$$2.1 = \frac{21}{10}$$

$$\text{SO } 2.1\overline{79} = \frac{21}{10} + \frac{79}{990} = \frac{21 \cdot 99 + 79}{990}$$

$$= \frac{1079}{495}$$

THIS IS GOOD FOR ME.

4. *(10 pts)*
Find the radius of convergence, and interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{4^n(x-2)^n}{3n}.$$

You are not required to determine whether the series is convergent at the endpoints of the interval of convergence.

FIRST, USE RATIO TEST TO GET RADIUS.

$$\lim_{n\to\infty} \left| \frac{4^{n+1}(x-2)^{n+1}}{3(n+1)} \cdot \frac{3n}{4^n(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{4(x-2)n}{n+1} \right|$$

$= |4(x-2)|$ WHICH IS LESS THAN 1 IF

$$-1 < 4(x-2) < 1$$
$$-\frac{1}{4} < x-2 < \frac{1}{4}$$
$$2-\tfrac{1}{4} < x < 2+\frac{1}{4}$$

$\boxed{\text{RADIUS OF CONV IS } 1/4}$

"INTERVAL OF CONVERGENCE" USUALLY MEANS CHECK ENDPOINTS, SO I WILL ANYWAY

IF $X = 2 + 1/4$, SERIES BECOMES $\sum_{n=1}^{\infty} \frac{1}{3n}$

WHICH IS $\frac{1}{3}$ OF A P-SERIES WITH $p=1$, SO DIV.

IF $x = 2 - 1/4$, SERIES BECOMES $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$

WHICH IS AN ALTERNATING SERIES.

TERMS ARE DECREASING, LIMIT $\lim_{n\to\infty} \left(\frac{1}{3n}\right) = 0$ SO CONV

$\boxed{\text{INTERVAL IS } [2-1/4,\ 2+1/4)}$