MAT 127 Solutions to Midterm 1 (03)

10 pts 2. Consider the sequence $\{a_n\}$ which begins

$$\frac{\sin(3)}{3}, \frac{\sin(5)}{9}, \frac{\sin(7)}{27}, \frac{\sin(9)}{81}, \frac{\sin(11)}{243}, \ldots$$

(a) Assuming the pattern continues, give an explicit formula a_n in the sequence above. Be sure to state the initial value of n.

Solution: Looking at the numerators, we can see that they are always the sine of an integer, and that these begin with 3 and increase by 2 each time. So if we start at $n = 0$, the numerators should be of the form $\sin(3 + 2n)$.

The denominators are all powers of 3, starting with the first. Since in this solution I chose to start with $n = 0$, that means they are of the form 3^{n+1}.

Thus, we have $\boxed{a_n = \frac{\sin(3+2n)}{3^{n+1}}}$. (If you choose to begin with $n = 1$, you'll instead have $\frac{\sin(1+2n)}{3^n}$. If you chose to start with $n = 47$, I'll leave it to you to adapt the answer.)

(b) Determine whether the sequence converges or diverges, fully justifying your answer. If it converges, give the limit it converges to.

Solution: The sequence $\boxed{\text{converges to zero}}$.

We can see this by noticing that $-1 \leq \sin(n) \leq 1$ for all values of n, and also that as $n \to \infty$, we have $\frac{1}{3^{n+1}} \to 0$. Hence $0 = \lim_{n\to\infty} \frac{-1}{3^{n+1}} \leq \lim_{n\to\infty} \frac{\sin(3+2n)}{3^{n+1}} \leq \lim_{n\to\infty} \frac{1}{3^{n+1}} = 0$, and then apply the squeeze theorem.

10 pts 3. Find the radius and interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n\sqrt{n}}(x-3)^n \quad .$$

Solution: To get the radius of convergence, we apply the ratio test and determine which values of x give us a ratio less than 1. We have

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{3^{n+1}\sqrt{n+1}} \cdot \frac{3^n\sqrt{n}}{(-1)^n(x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{x-3}{3}\sqrt{\frac{n}{n+1}} \right| = \left| \frac{x-3}{3} \right| \quad .$$

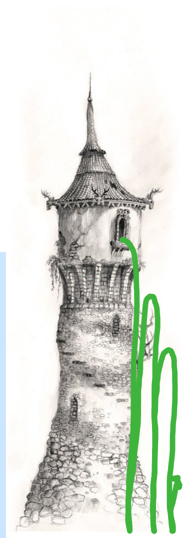
This ratio will be less that 1 when $|x - 3| < 3$, so $\boxed{\text{the radius of convergence is 3}}$ and the center is 3, that is, we have convergence for $0 < x < 6$.

Now we check the endpoints to get the interval of convergence.

- When $x = 0$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$. This diverges, since it is a p-series with $p = \frac{1}{2}$.
- When $x = 6$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}} \frac{(3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This is an alternating series. Since $1/\sqrt{n} < 1/\sqrt{n+1}$, the absolute values of the terms are decreasing, and also $\lim_{n\to\infty}(1/\sqrt{n}) = 0$. Hence, the alternating series test tells us that this is a convergent series.

Consequently, $\boxed{\text{the interval of convergence is } (0, 6]}$.

10 pts 4. The fair princess Aracelushka is being held prisoner by the evil monkey-king. As a signal to her brother Jack, she drops an enchanted orb out of the window of the tower where she is being held, 120 feet above the ground. Each time the orb strikes the ground, it sends out a beacon of golden light, then bounces and returns to a height two-thirds of its previous maximum height. What is the total vertical distance traveled by the orb if it bounces infinitely many times?



Solution: Since each bounce is $2/3$ of the previous, we have a geometric series of ratio $r = 2/3$.
However, keep in mind that the solution is *not*

$$120 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{120}{1 - 2/3} = 360 \text{ feet.}$$

That represents the total distance the orb travels downwards. But also the orb travels up again that same distance (except for the first downward fall of 120 feet). So the total distance traveled is twice the above answer, minus the height of the tower window. That is, the orb travels

$$2 \cdot 120 \left(\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \right) - 120 = 240 \cdot 3 - 120 = \boxed{600 \text{ feet}} .$$

5. Let $f(x) = \dfrac{2}{1+4x^3}$.

5 pts (a) Write the Maclaurin series (ie, the Taylor series about 0) for $f(x)$.

Solution: We start with the geometric series $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots$, and substitute $z = -4x^3$, then multiply by 2 to obtain the series

$$\frac{2}{1+4x^3} = 2\sum_{n=0}^{\infty}(-1)^n\, 4^n x^{3n} = 2 - 8x^3 + 32x^6 - 128x^9 + \ldots$$

5 pts (b) Write the Maclaurin series for $\displaystyle\int f(x)\,dx$.

Solution: Just integrate the previous answer with respect to x to obtain

$$\int f(x)\,dx = C + 2\sum_{n=0}^{\infty}\frac{(-1)^n\, 4^n x^{3n+1}}{3n+1} = C + 2x - 2x^4 + \tfrac{32}{7}x^7 - \ldots$$

5 pts (c) Would it be reasonable to use the first few nonzero terms from your answer to the previous part to estimate $\int_0^1 f(x)\,dx$. Why or why not? Explain. (You don't have to estimate this value, just explain why or why not it would make sense to try.)

Solution: No, it would not. Since the geometric series converges for $|z| < 1$, after the substitution we have convergence for $|4x^3| < 1$, that is $|x| < \dfrac{1}{\sqrt[3]{4}}$, which is less than 1. Since the series does not converge for $x = 1$, it would not be reasonable to use it as an approximation of the integral.

15 pts 6. For each of the infinite series below, determine whether they converge or diverge. State explicitly how you determined this, and justify your answers fully.

(a) $\displaystyle\sum_{n=0}^{\infty} n\pi$

Solution: Since the sum is $\pi + 2\pi + 3\pi + \ldots$, this certainly diverges. More officially, since $\lim_{n\to\infty} n\pi \neq 0$, the series diverges by the divergence test.

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$

Solution: This is a decreasing sequence of positive terms which agrees with a decreasing continuous function, so we can apply the integral test.

We compute $\int_2^\infty \frac{dx}{x(\ln(x))^2}$ by first making the substitution $u = \ln x$ so $du = dx/x$. Then we have

$$\int_{\ln 2}^\infty \frac{du}{u^2} = \lim_{M\to\infty} \left.\frac{-1}{u}\right|_{\ln 2}^M = \lim_{M\to\infty}\left(\frac{-1}{M} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Since the improper integral converges, $\boxed{\text{the series converges}}$.

(c) $\displaystyle\sum_{n=0}^\infty \frac{n^3+n}{n^5-n^3+2}$

Solution: Using the limit comparison test with the convergent p-series $\sum \frac{1}{n^2}$, we have

$$\lim_{n\to\infty} \frac{n^3+n}{n^5-n^3+2}\cdot\frac{1}{1/n^2} = \lim_{n\to\infty}\frac{n^5+n^2}{n^5-n^3+2} = 1$$

so the series are comparable and both series $\boxed{\text{converge}}$.

If you prefer to use direct comparison to $\sum \frac{1}{n^2}$, you can't, since $\frac{n^3+n}{n^5-n^3+2} > \frac{1}{n^2}$. However, we can compare with $\sum \frac{2}{n^2}$, since

$$\frac{n^3+n}{n^5-n^3+2} < \frac{2}{n^2} \quad\iff\quad n^5+n^3 < 2n^5-2n^3+4$$

which holds for all positive n. So we can conclude that the series converges.

If you try to use the ratio test, you will get no information because the limit of the ratio is 1.

If you try to use the integral test, probably you will cry a lot. The corresponding integral is not easy to do at all, although WolframAlpha tells me it is about 1.36118 or so.