EXERCISE ONE Determine the radius of convergence and the interval of convergence (you have to determine whether the end points of the interval are contained in the interval) of the power series

$$\sum_{n=0}^{+\infty} \frac{n^2}{e^n} x^n.$$

Denote by a_n the term $\frac{n^2}{e^n} x^n$. The ratio $\frac{a_{n+1}}{a_n}$ is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n^2} \frac{e^n}{e^{n+1}} \frac{x^{n+1}}{x^n} = \frac{(n+1)^2}{n^2} \frac{x}{e}.$$

The limit of the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is thus $\frac{x}{e}$. If $|x| < e$ then $|\frac{x}{e}| < 1$ and the series $\sum_{n=0}^{+\infty} \frac{n^2}{e^n} x^n$ is convergent by the Ratio Test. If $|x| > e$ then the series is divergent by the Ratio Test. Therefore the radius of convergence is e. If $x = e$ then we get the series $\sum_{n=0}^{+\infty} n^2$. Its term sequence $\{n^2\}$ is divergent. By Divergence Test $\sum_{n=0}^{+\infty} n^2$ is a divergent series. Similarly if $x = -e$ then we get a divergent series as well. Therefore the interval of convergence is $(-e, e)$.