EXERCISE ONE Let $\{a_n\}_{n=1}^{+\infty}$ be a sequence of real numbers defined by the following recurrence relation:

- $a_1 = 3$;
- $a_{n+1} = \frac{a_n}{2} + \frac{1}{2}$ for all n.

Let $\{b_n\}_{n=1}^{+\infty}$ be a sequence such that $b_n = a_n - 1$ for all n. Prove that b_n is a geometric sequence and find an explicit formula for $\{b_n\}$.

We write the equation $a_{n+1} = \frac{a_n}{2} + \frac{1}{2}$ as $b_{n+1} + 1 = \frac{b_n+1}{2} + \frac{1}{2}$. The latter equation can be simplified as $b_{n+1} = \frac{b_n}{2}$. This means that $\{b_n\}$ is a geometric sequence with ratio $\frac{1}{2}$. Thus $b_n = c\frac{1}{2^n}$ for some constant c. To determine c, we use $b_1 = a_1 - 1 = 2$. Since $b_1 = c\frac{1}{2}$, we have $c = 2b_1 = 4$. Therefore $b_n = \frac{4}{2^n}$.

EXERCISE TWO Consider the sequence $\{\frac{126e^{\frac{1}{n}}+1}{\cos(\frac{1}{n^2})}\}$. Does it converge? If yes what is the limit?

We know that the sequence $\{\frac{1}{n}\}$ converges to 0. By applying Theorem II about continuous functions to e^x, we obtain that $\lim e^{\frac{1}{n}} = e^0 = 1$. The same theorem applied to the sequence $\frac{1}{n^2}$ and the function cos implies that $\lim\cos(\frac{1}{n^2}) = \cos(0) = 1$. By the theorem about algebraic operations and limits, we conclude that the sequence converges and the limit is $\frac{126\lim e^{\frac{1}{n}}+1}{\lim\cos(\frac{1}{n^2})} = \frac{126+1}{1} = 127$.