EXERCISE 1

(1) The series is a p-series with $p = 3 > 1$, thus converges.

(2) The series is a geometric series with ratio $\frac{5}{4} > 1$, thus diverges.

(3) Any solution of this differential equation is an antiderivative of the constant function 1. Thus the general solution is $x + C$ with $C \in \mathbf{R}$. The initial condition can be written as $3 = 2 + C$. Thus $C = 1$ and the desired particular solution is the function $x + 1$.

(4) The derivative of y is e^x. Thus $y - y' = 1$ is a desired differential equation. (The solution of this question is not unique, you can find other equations.)

EXERCISE 2 The term sequence $(\frac{(-9)^{n-1}}{10^n})$ can be written as

$$(\frac{-1}{9}(\frac{(-9)}{10})^n).$$

Thus the series is a geometric series with ratio $\frac{-9}{10}$ and initial term $\frac{1}{10}$. The sum is

$$\frac{\frac{1}{10}}{1 - \frac{-9}{10}} = \frac{1}{19}.$$

EXERCISE 3 Since $0 \leq \cos^2(n) \leq 1$ and $n^2 + 1 > n^2$, we have

$$0 \leq \frac{\cos^2(n)}{n^2+1} < \frac{1}{n^2}.$$

By the comparison test the series converges because the p-seris $\sum \frac{1}{n^2}$ converges.

EXERCISE 4 Since n and n^2+1 are positive, $\frac{n}{n^2+1}$ is postive for all n. Therefore the series is an alternating series. We apply the alternating series test. We need to show that the sequence $(\frac{n}{n^2+1})$ is decreasing and converges to 0.

Let k be a positive integer. We have

$$(k+1+\frac{1}{k+1}) - (k+\frac{1}{k}) = 1 - \frac{1}{k(k+1)} > 0.$$

Therefore

$$\frac{k+1}{(k+1)^2+1} = \frac{1}{k+1+\frac{1}{k+1}} < \frac{1}{k+\frac{1}{k}} = \frac{k}{1+k^2},$$

i.e. the sequence $(\frac{n}{n^2+1})$ is decreasing.

Since

$$0 < \frac{n}{n^2+1} = \frac{1}{n+\frac{1}{n}} < \frac{1}{n}$$

and $\lim \frac{1}{n} = 0$, we have $\lim \frac{n}{n^2+1} = 0$ by Squeeze Theorem.

EXERCISE 5

This is a power series centered at -1 with positive coefficients. We apply the ratio test. We have

$$\lim \frac{\frac{n+1}{4^{n+1}}}{\frac{n}{4^n}} = \lim \frac{n+1}{4n} = \frac{1}{4}.$$

Thus the radius of convergence is 4. The two end points of the interval of convergence are $-1-4 = -5$ and $-1+4 = 3$.

At 3 we have the numerical series $\sum_{n=1}^{\infty} n$. It is divergent by the divergence test because the term sequence (n) does not converge to 0. At -5 we have the series $\sum_{n=1}^{\infty} (-1)^n n$. Again it is divergent by the divergence test.

Hence the interval of convergence does not contain the two end points and is equal to the open interval $(-5, 3)$.

EXERCISE 6

This is a separable equation. We write it as $y'y = x$. The antiderivatives of the right-hand side are $\frac{x^2}{2} + C$ for $C \in \mathbf{R}$. The function $\frac{y(x)^2}{2}$ is an antiderivative of the left-hand side with respect to the variable x. Thus

$$\frac{y(x)^2}{2} = \frac{x^2}{2} + C$$

for some C. The initial condition can be written as $(-3)^2/2 = 0 + C$. Thus $C = \frac{9}{2}$.

Either $y = \sqrt{x^2+9}$ or $y = -\sqrt{x^2+9}$. Since $y(0) = -3$ is negative, our solution should be

$$y = -\sqrt{x^2+9}.$$

EXERCISE 7

We denote by $A(x)$ the value obtained by the Euler method at a point x. The first order Taylor polynomial of y at 0 is

$$y(0) + y'(0)x = 1 + (y(0) + 0y(0))x = 1 + x.$$

Thus

$$A(0.2) = 1 + 0.2 = 1.2.$$

The first order Taylor polynomial of y at 0.2 is

$$y(0.2) + y'(0.2)x = y(0.2) + (y(0.2) + 0.2y(0.2))x.$$

Thus

$$A(0.4) = A(0.2) + (A(0.2) + 0.2A(0.2))0.2 = 1.2 + 1.44 \times 0.2 = 1.488.$$

EXERCISE 8 Denote by a the initial amount of money. Let $r = 0.05$. Then at the end of the n-th year, the amount of money is

$$a(1+r)^n.$$

We want $(1+r)^n = 3$, i.e. $n = \log_{1.05}(3)$. However n should be an integer. So we take $\lceil \log_{1.05}(3) \rceil$, the least integer larger than $\log_{1.05}(3)$.

EXERCISE 9 We denote by P the population and by t the time in hours. The exercise told us that $P' = rP$ for some constant r. The general solution of this equation is $e^{rt} + C$ with $C \in \mathbf{R}$. We have $P(0) = C = 100$. The condition $P(1) = 400$ can be written as $e^r + 100 = 400$. Thus $e^r = 300$ and our solution is $300e^t + 100$.

EXERCISE 10 We denote by u the amount of salt in kg in the tank and by t the time in minutes. The income is

$$0.05 \times 5kg/min = 0.25kg/min.$$

The concentration of salt in the tank at time t is $u/1000$. The outcome is thus

$$5u/1000kg/min = u/200kg/min.$$

Hence the differential equation satisfied by u is

$$u' = 0.25 - u/200.$$

We write it as

$$\frac{u'}{0.25 - u/200} = 1.$$

By integrating both sides, we get

$$\ln|u/200 - 0.25| = x/200 + C$$

for some C. Therefore

$$u = 50 \pm 200e^{x/200+C}.$$

As $u(0) = 0$, our solution has the form $50 - 200e^{x/200+C}$. Moreover $50 - 200e^C = 0$ implies that $e^C = 0.25$. Hence our solution is

$$u = 50 - 50e^{x/200}.$$