EXERCISE 1

(1) Consider the sequence $(|a_n|)$. As $|\cos| \leq 1$, we have
$$0 \leq |a_n| \leq \frac{2^n+1}{5^n}.$$
As $\frac{2^n+1}{5^n} = \frac{1+\frac{1}{2^n}}{(\frac{5}{2})^n}$, $\lim \frac{1}{2^n} = 0$ and $\lim(\frac{5}{2})^n = +\infty$, we get
$$\lim \frac{2^n+1}{5^n} = 0.$$
By Squeeze Theorem we obtain $\lim |a_n| = 0$. A sequence converges to zero if and only if its absolute value sequence converges to zero. Therefore the limit of (a_n) is 0.

(2) We divide both numerator and denominator of a_n by n^3 and get $a_n = \frac{1+2/n+4/n^2}{2+2/n^2}$. Since $1/n$ and $1/n^2$ converges to 0, the sequence (a_n) converges to $\frac{1}{2}$.

(3) Consider the function $f(x) = \frac{\ln(x)+1}{x^{1/3}}$. The derivative of $\ln(x)+1$ is $\frac{1}{x}$ and the derivative of $x^{1/3}$ is $\frac{x^{-2/3}}{3}$. By l'Hôpital's rule, we have
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{x^{-2/3}}{3}} = \lim_{x\to+\infty} 3x^{-1/3} = 0.$$
By a theorem that we learned in MAT127, the limit of (a_n) is 0.

EXERCISE 2

(1) First remark that for any positive integer n the number n^2+2n-2 is positive. This means that we have an alternating series. We want to apply the alternating series test to prove that the series is convergent. Let $b_n = \frac{1}{n^2+2n-2}$. We need to prove that $\lim b_n = 0$ and that the sequence (b_n) is decreasing. The decreasing property follows from the fact that the denominators n^2+2n-2 form an increasing sequence. It is clear that the limit of (b_n) is 0.

(2) We already did this question this semester. Integral test.

(3) We denote by (a_n) the term sequence of the given series. We want to prove the convergence of the series by proving its absolute convergence. Since $|\cos|$ is always ≤ 1, we have $|a_n| \leq \frac{2^n+1}{5^n}$. Since the series $\sum \frac{2^n}{5^n}, \sum \frac{1}{5^n}$ are geometric series with postive ratio < 1, they converge. Therefore the sum of these two series, namely $\sum \frac{2^n+1}{5^n}$ converges as well. By the comparison test, the series $\sum |a_n|$ is convergent. QED.

EXERCISE 3

(1) The McLaurin series of $\ln(1+x)$ is
$$\sum_{n=0}^{\infty} (-1)^{n+1} x^n/n.$$
The substituion $x \mapsto x^2/2$ gives us
$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x^2/2)^n/n = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n2^n}.$$
Hence
$$f(x) = x^2 \ln(1+x) = x^2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n2^n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+2}}{n2^n}.$$

(2) We apply the ratio test. Denote by a_n the term $(-1)^{n+1}\frac{x^{2n+2}}{n2^n}$. We have
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2x^2 n}{n+1}.$$
This ratio converges to $2x^2$. For $2x^2 < 1$, we need $|x| < \frac{\sqrt{2}}{2}$. Thus the radius is $\frac{\sqrt{2}}{2}$.

(3) We integrate the above series term by term. The function $(-1)^{n+1}\frac{x^{2n+3}}{(2n+3)n2^n}$ is a primitive function of $(-1)^{n+1}\frac{x^{2n+2}}{n2^n}$. By the fundamental theorem of calculus we obtain

$$\int_0^1 (-1)^{n+1}\frac{x^{2n+2}}{n2^n}dx = (-1)^{n+1}\frac{1}{(2n+3)n2^n}.$$

Therefore

$$\int_0^1 f(x)dx = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+3)n2^n}.$$

EXERCISE 4

(1) The equation can be written in a separable form as follows :

$$y'/(1+y) = x^2.$$

The antiderivatives of the right-hand side are $x^3/3 + C$ with $C \in \mathbf{R}$. The function $\ln|1+y|$ is an antiderivative of the left-hand side. Therefore

$$\ln|1+y| = x^3/3 + C$$

for some C. By taking exponential of both sides we get

$$|y+1| = e^{x^3/3+C}.$$

Either $y = -1 + e^{x^3/3+C}$ or $y = -1 - e^{x^3/3+C}$. The initial condition $y(0) = 1$ is > -1, thus $y = -1 + e^{x^3/3+C}$. Moreover the initial condition gives us $-1 + e^C = 1$, i.e. $e^C = 2$. Hence our desired solution is $-1 + 2e^{x^3/3}$.

(2) The characteristic equation is $z^2 + 2z + 1 = 0$. It has a double real root -1. The general solution of the differential equation is $C_1e^{-x} + C_2xe^{-x}$. The initial conditions now can be written as $y(0) = C_1 = 2$ and $y'(0) = -C_1 + C_2 = 2$. Therefore $C_1 = 2, C_2 = 4$ and our desired solution is $2e^{-x} + 4xe^{-x}$.

EXERCISE 5

We denote by $A(x)$ the value obtained by the Euler method at a point x. The first order Taylor polynomial of y at 0 is

$$y(0) + y'(0)x = 1 + (0 + y(0))x = 1 + x.$$

Thus

$$A(0.1) = 1 + 0.1 = 1.1.$$

The first order Taylor polynomial of y at 0.1 is

$$y(0.2) + y'(0.2)x = y(0.1) + (2 \times 0.1 + y(0.1))x.$$

Thus

$$A(0.2) = A(0.1) + (0.2 + A(0.1))0.1 = 1.1 + 1.3 \times 0.1 = 1.23$$

The first order Taylor polynomial of y at 0.2 is

$$y(0.2) + y'(0.2)x = y(0.2) + (2 \times 0.2 + y(0.2))x.$$

Thus

$$A(0.3) = A(0.2) + (0.4 + A(0.2))0.1 = 1.23 + 1.63 \times 0.1 = 1.393.$$

EXERCISE 6 These topics are not covered in MAT127 fall 2021.

EXERCISE 7 We denote by P the population of bees and by t the number of years. Since the initial population is below the carrying capacity, our solution of the logistic equation has the form

$$P(t) = \frac{KP(0)e^{rt}}{K - P(0) + P(0)e^{rt}}$$

where $K = 10000$ is the carrying capacity, $P(0) = 1000$ is the initial population and r is some constant. The population after two years is

$$2000 = P(2) = \frac{10000 \times 1000e^{2r}}{9000 + 1000e^{2r}}.$$

This equation gives us $e^{2r} = \frac{9}{4}$. The population after four years is

$$P(4) = \frac{10000 \times 1000e^{4r}}{9000 + 1000e^{4r}} = \frac{10000 \times 1000(\frac{9}{4})^2}{9000 + 1000(\frac{9}{4})^2} = 3600.$$