Calculus C, MAT127

Fall 2017

Final Exam

Name: SOLUTIONS

ID Number:________________

PLEASE CIRCLE YOUR SECTION:

LEC 01, MWF 10:00-10:53am LEC 02, MF 1:00-2:20pm LEC 04, TuTh 5:30-6:50pm

This is a closed book, closed notes test. No consultations with others. Calculators are not allowed.

Please turn off and take off the desk cell phones and any other electronic devices. Only the exam and pens/pencils should be on your desk. If you need extra paper, ask your proctors.

Please explain all your answers and SHOW ALL WORK, unless the question says otherwise. Answers without explanation will receive little credit.

The problems are *not* in the order of difficulty. You may want to look through the exam and do the easier questions first.

Each question is worth 20 points.

DO NOT TURN THIS PAGE UNTIL INSTRUCTED TO DO SO

Please do not write in this table

1	2	3	4	5	6	7	Total
20 pts	20 pts	20 pts	20 pts	20 pts	20 pts	20 pts	140 pts

Reference Page

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

The Taylor series formula for $f(x)$ around a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

The Taylor inequality for the remainder $R_n = f(x) - \left[f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n\right]$:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1},$$

if $|f^{(n+1)}(t)| \le M$ for all t between a and x.

1. Determine whether the following series converge or diverge. Justify your answers.

(a) $\sum_{n=0}^{\infty}(-1)^n \frac{(n+1)^7}{4^{3n-2}}$ $16 - 32 + \frac{2187}{256} - 1 + \frac{78125}{1048576} - \cdots$

ALTERNATING SERIES.

- TERMS DECREASE IN ABS. VALUE
- $\lim_{n\to\infty} \frac{(n+1)^7}{4^{3n-2}} = 0$ (SINCE FOR n LARGE, $4^{3n} >> n^7$.)

So $\boxed{\text{CONVERGES BY ALT. SERIES TEST}}$

(b) $\sum_{n=1}^{\infty}(e^{-3n} + n^2)$ $= (e^{-3}+1) + (e^{-6}+4) + (e^{-9}+9) + \cdots$

$\lim_{n\to\infty}\left(e^{-3n} + n^2\right) = +\infty$ ($e^{-3n} \to 0$, BUT $n^2 \to \infty$)

$\boxed{\text{DIVERGES BY DIVERGENCE TEST}}$

(c) $\sum_{n=0}^{\infty} \frac{2^{n+1}+(-5)^n}{2^{2n+1}}$ $= \frac{3}{2} - \frac{1}{8} + \frac{33}{32} - \frac{109}{128} + \frac{657}{512} - \frac{3061}{2048} + \cdots$

ALTERNATING, BUT n^{TH} TERM DOES NOT TEND TO 0. CAN REWRITE AS

$\sum\left(\frac{1}{2^n} + \frac{(-1)^n 5^n}{2^{2n+1}}\right)$. WHILE $\lim_{n\to\infty} \frac{1}{2^n} = 0$

$\lim_{n\to\infty} \frac{5^n}{2^{2n+1}} = \lim \frac{1}{2}\left(\frac{5}{4}\right)^n = +\infty$.

$\boxed{\text{SO THE SERIES DIVERGES}}$

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2-1}{n^6+n}$ $= 0 + \frac{1}{22} - \frac{2}{183} + \frac{3}{820} - \frac{4}{2605} + \cdots$

ALTERNATING SERIES.

- ABS VALUE DECREASES.
- $\lim_{n\to\infty} \frac{n^2-1}{n^6+n} = \lim_{n\to\infty} \frac{n^2}{n^6} = 0$

$\boxed{\text{SERIES CONVERGES BY ALT. SERIES TEST}}$

(e) For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-7x)^n}{4^n}$ converge? For these values of x, find the sum of the series (as a function of x).

OBSERVE THIS IS "GEOMETRIC" WITH RATIO $\left(\frac{-7x}{4}\right)$.

SO WHEN IT CONVERGES, THE SUM IS

$$\frac{1}{1-\left(\frac{-7x}{4}\right)} = \frac{4}{4+7x}$$

IT WILL CONVERGE WHEN $\left|\frac{-7x}{4}\right| < 1$ (AND DIVERGE FOR $\left|\frac{-7x}{4}\right| \geq 1$).. YOU COULD USE THE RATIO TEST AND CHECK ENDPOINTS, OR JUST KNOW ABOUT GEOM. SERIES).

SO $\left|\frac{-7x}{4}\right| < 1 \iff |x| < 4/7$

$\boxed{\text{CONVERGES FOR } -4/7 < x < 4/7}$

AND DIVERGES OTHERWISE

2. Consider the sequence $\{a_n\}_{n=1}^{\infty}$, $a_n = \dfrac{n}{n^2+1}$. $= \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \ldots$

(a) Does the sequence $\{a_n\}$ converge? If so, find its limit. Show all work and explain your answer.

$\boxed{\text{CONVERGES}}$ SINCE $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2} = \lim \frac{1}{n} = 0$

(b) Does the sequence $\{(-1)^n a_n\}$ converge? If so, find its limit. Explain your answer.

YES, SINCE THE LIMIT IS $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

(continued) We are working with the sequence $\{a_n\}_{n=1}^{\infty}$, $a_n = \dfrac{n}{n^2+1}$.

(c) Is the sequence $\{a_n\}$ increasing, decreasing, or not monotonic? Justify your answer. (For full credit, carefully compare a_n and a_{n+1}.)

COMPARING: $a_n > a_{n+1}$ IF $\dfrac{n}{n^2+1} > \dfrac{n+1}{(n+1)^2+1}$

$\iff n(n^2+2n+2) > (n^2+1)(n+1)$

$\iff n^3+2n^2+2n > n^3+n^2+n+1$

TRUE FOR $n \geq 1$.

SEQUENCE IS MONOTONE DECREASING

(d) Does the series $\sum_{n=1}^{\infty}(-1)^n a_n$ converge? Justify your answer.

YES

IT CONVERGES BY THE ALTERNATING SERIES TEST.

- $a_n \geq 0$
- $\{a_n\}$ IS DECREASING
- $\lim_{n\to\infty} a_n = 0$.

3. (a) For the function $f(x) = \ln(1-3x)$, find the Taylor series centered at 0.

SINCE WE MAY HAVE FORGOTTEN THE MACLAURIN SERIES FOR $\ln(1+y)$, LET'S REMEMBER: $\left(\frac{1}{1-x} = \sum x^n\right)$

$$\ln(1+y) = \int \frac{dy}{1+y} = \int \sum_{n=0}^{\infty} (-y)^n \, dy = \int (1 - y + y^2 - y^3 + \cdots)\, dy$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n y^{n+1}}{n+1} = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$$

LET $y = -3x$ TO GET

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-3x)^{n+1}}{n+1} = \boxed{-\sum_{n=0}^{\infty} \frac{3^{n+1}}{n+1} x^{n+1} = -3x - \frac{3^2}{2}x^2 - \frac{3^3}{3}x^3 - \cdots}$$

(b) Find the interval of convergence of the series you found in (a). OR $-\sum_{n=1}^{\infty} \frac{3^n}{n} x^n$

WE CAN "KNOW" THIS CONVERGES FOR $|3x| < 1$ BECAUSE IT IS THE INTEGRAL OF A GEOMETRIC SERIES WITH RATIO $-3x$. OR USE THE RATIO TEST:

$$\lim_{n\to\infty} \left| \frac{(3x)^{n+2}}{n+2} \cdot \frac{n+1}{(3x)^{n+1}} \right| = \lim_{n\to\infty} \left| 3x \, \frac{n+1}{n+2} \right| = |3x|,$$ SO CONV FOR $|x| < \frac{1}{3}$.

NOW WE MUST CHECK $|x| = \frac{1}{8}$:

- IF $x = \frac{1}{3}$, SERIES IS $-\sum_{n=1}^{\infty} \frac{1}{n}$, WHICH DIVERGES (HARMONIC, OR P-SERIES $P=1$)
- IF $x = \frac{-1}{3}$, SERIES IS $-\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, WHICH CONVERGES BY THE ALT. SERIES TEST (DECREASING, LIMIT IS 0)

SO,

$$\boxed{\text{INTERVAL OF CONVERGENCE IS } \frac{-1}{3} \le x < \frac{1}{3}}$$

(c) For the function $g(x) = \sqrt{x}$, find the first 5 terms of the Taylor series centered at 1.
(If your answer includes expressions such as 5^3 or $4!$, leave them in this form; do not compute.)

WE COULD USE THE BINOMIAL SERIES TO GET $(x-1)^{1/2}$, BUT LET'S DO IT THE LONG WAY.

THE FIRST 5 TERMS MEANS WE NEED THE 4TH DERIVATIVE.

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$
0	$x^{1/2}$	1	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$	$1/2$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$	$\frac{1}{4\cdot 2!} = \frac{1}{2^2\cdot 2!}$
3	$\frac{3}{8}x^{-5/2}$	$3/8$	$\frac{3}{8\cdot 3!} = \frac{3}{2^3\cdot 3!}$
4	$\frac{-15}{16}x^{-7/2}$	$\frac{-15}{16}$	$\frac{15\cdot}{16\cdot 4!} = \frac{5\cdot 3}{2^4\, 4!}$

SO THE SERIES IS

$$\boxed{\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{(x-1)^2}{2^2\cdot 2!} + \frac{3(x-1)^3}{2^3\cdot 3!} - \frac{3\cdot 5(x-1)^4}{2^4\cdot 4!} + \cdots}$$

4. Use series to answer the following questions.

(a) Estimate $\cos(0.1)$ correct to 3 decimal places. Explain how you know that your 3 decimal places are guaranteed to be correct.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

SO

$$\cos(.1) = 1 - \frac{.01}{2} + \frac{.0001}{24} - \cdots$$

$$\approx 0.995$$

THIS IS GOOD TO 3 PLACES, SINCE THE ERROR IS NO MORE THAN THE MAX OF $\frac{x^4}{4!}$ FOR $0 < x < 0.01$, WHICH IS $\frac{.0001}{24}$ (BY TAYLOR'S THEOREM)

(b) Compute $\displaystyle\lim_{x\to 0} \frac{\cos 2x - e^{-x^2} + x^2}{x^3 \sin x}$.

$$\lim_{x\to 0} \frac{\cos 2x - e^{-x^2} + x^2}{x^3 \sin x}$$

$$= \lim_{x\to 0} \frac{\left(\cancel{1} - \cancel{\frac{(2x)^2}{2}} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right) - \left(\cancel{1} + \cancel{(-x^2)} + \frac{(-x^2)^3}{3!} + \cdots\right) + \cancel{x^2}}{x^3\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{16x^4}{4!} - \frac{64x^6}{6!} - \frac{x^6}{3!} + \cdots}{x^4\left(1 - \frac{x^2}{3!} + \frac{x^5}{5!} + \cdots\right)} = \lim_{x\to 0} \frac{\frac{16}{4!} - \overbrace{\left(\frac{64}{6!} + \frac{1}{3!}\right)x^6 + \cdots}^{\to 0}}{1 + \underbrace{\frac{x^2}{3!} + \cdots}_{\to 0}}$$

$$= \frac{16}{4!} = \frac{4}{3\cdot 2} = \boxed{\frac{2}{3}}$$

(c) For the initial value problem $y' = xy + 1$, $y(0) = 2$, find solution of the form

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots.$$

Give the first 5 terms of the series only; you do not have to find other coefficients or describe a pattern.

SINCE $y(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \cdots$

(1) $y'(x) = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + 5C_5 x^5 + \cdots$

$=$

$+\begin{cases} xy = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^5 + \cdots \\ 1 = 1 \end{cases}$

SINCE $y(0) = 2 = C_0 + 0 + 0 + 0$

$\boxed{C_0 = 2}$

THIS SAYS THAT $\boxed{C_1 = 1}$

$2C_2 = C_0$, SO $\boxed{C_2 = 1}$

$3C_3 = C_1$, SO $\boxed{C_3 = 1/3}$

$4C_4 = C_2$, SO $\boxed{C_4 = 1/4}$

WE DON'T NEED IT, BUT $5C_4 = C_3$

SO $C_4 = 1/15$.

$$\boxed{y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots}$$

5. Solve the following differential equations.

(a) Solve the equation $y' = (t^2 + t + 1)(y + 1)$. WE GET $\boxed{y=-1}$ RIGHT AWAY.

IF $y \neq -1$, SEPARATE TO GET

$$\int \frac{dy}{y+1} = \int (t^2+t+1)\,dt$$

$$\ln|y+1| = \frac{t^3}{3} + \frac{t^2}{2} + t + C$$

EXPONENTIATE:

$$|y+1| = e^{t^3/3 + t^2/2 + t + C}$$

$$y = 1 \pm e^{t^3/3 + t^2/2 + t + C} = \boxed{1 + A e^{t^3/3 + t^2/2 + t}}$$

WITH A ANY NUMBER POSITIVE, NEGATIVE OR 0.

(b) Solve the initial value problem $y' = xe^{x^2}y^2$, $y(1) = 2$.

ASSUME $y \neq 0$, SO

$$\int \frac{dy}{y^2} = \int x e^{x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{x^2} + C$$

LET $u = x^2$, $du = 2x\,dx$

$$-\frac{1}{y} = \frac{1}{2}e^{x^2} + C$$

SO

$$y = \frac{-1}{\frac{1}{2}e^{x^2} + C} = \frac{-2}{e^{x^2} + C_1}$$

SINCE $y(1) = 2$,

$$2 = \frac{-2}{e + C_1}$$

$$e + C_1 = \frac{-2}{2} = -1$$

$$C_1 = -1 - e$$

$$\boxed{y(x) = \frac{-2}{e^{x^2} - 1 - e}}$$

WE DIDN'T COVER THIS. I ASSUME ITS A TYPO, AND IT MEANT $y'' - 6y' + 9y = 0$

(c) Solve the initial value problem $\boxed{y'' - 6y' + 9 = 0,}$ $y(0) = 1$, $y'(0) = 2$.

CHAR POLY $r^2 - 6r + 9 = (r-3)^2$, ZERO IF $r = 3$.

SO GENERAL SOLUTION IS $y(x) = e^{3x}(Ax+B)$

$y(0)=1$ SAYS $\boxed{1 = B}$

$y'(x) = 3e^{3x}(Ax+B) + e^{3x}(A)$

$y'(0) = 3B + A = 2$, SO $\boxed{A = -1}$

$$\boxed{y(x) = e^{3x}(1-x) = e^{3x} - xe^{3x}}$$

Now, check (by differentiation) that your solution indeed satisfies the equation.

$y' = 3e^{3x}(1-x) - e^{3x} = 2e^{3x} - 3xe^{3x}$

$y'' = 6e^{3x} - 3e^{3x} - 9xe^{3x} = 3e^{3x} - 9xe^{3x}$

SO

$y'' - 6y' + 9y$

$= (3e^{3x} - 9xe^{3x}) - (12e^{3x} - 18xe^{3x}) + (9e^{3x} - 9xe^{3x}) = 0$

THE QUESTION AS WRITTEN: $y'' - 6y' + 9 = 0$

FIRST, SOLVE $y'' - 6y' = 0 \Rightarrow r = 6$ OR $r = 0$

$y = Ae^{6x} + B$

NOW IF $-6y' = -9$, $y' = 3/2 \Rightarrow y = \frac{3x}{2} + C$

SO $y = Ae^{6x} + \frac{3x}{2} + C$ IS THE GENERAL SOL'N.

THEN DETERMINE A & C ...

(d) Solve the equation $2y'' + 2y' + \cancel{1}\,y = 0$.

CHAR POLY IS

$$2r^2 + 2r + 1 = 0$$

$$r = \frac{-2 \pm \sqrt{4-8}}{4} = \frac{-2 \pm 2i}{4} = -\frac{1}{2} \pm \frac{1}{2} i$$

SO GENERAL SOLUTION IS

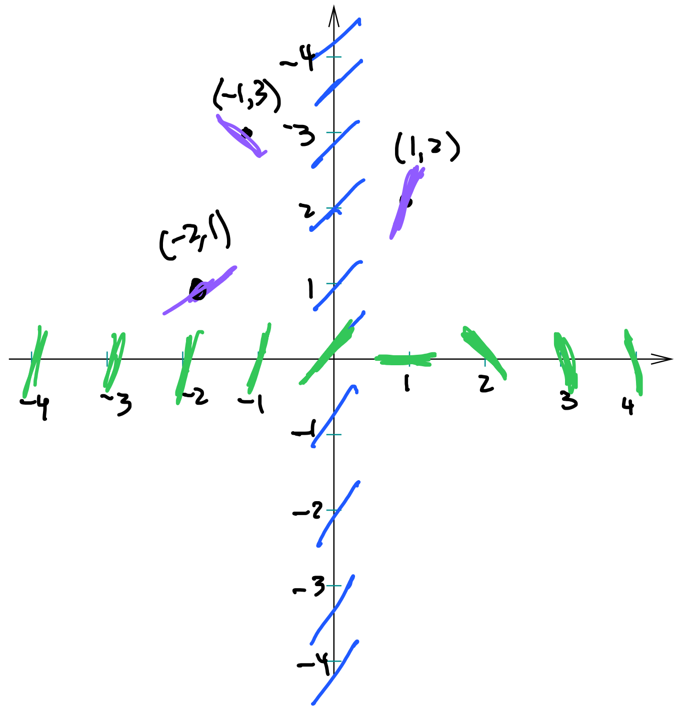
$$\boxed{y(t) = e^{-t/2}\left(A\cos(t/2) + B\sin(t/2)\right)}$$

6. In this question, you will use the direction field approach to study the equation

$$y' = xy - x + 1.$$

Do not solve this equation.

(a) Sketch the direction field along the x-axis, along the y-axis, and at points $(1, 2)$, $(-2, 1)$, $(-1, 3)$. Use the grid below. Please label the axes and all relevant points.



IF $X=0$, $y'=1$

IF $y=0$, $y'=1-X$

AT $(1,2)$
$y'=2-1+1=2$

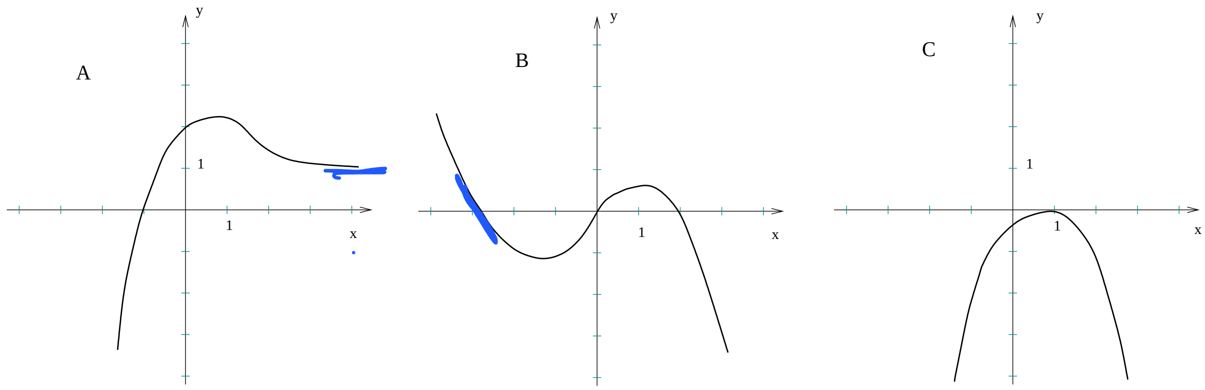
AT $(-2,1)$
$y'=-2+2+1=1$

AT $(-1,3)$
$y'=-3+1+1=-1$

(b) Consider the solution $y(x)$ of the initial value problem $y' = xy - x + 1$, $y(2) = 0$. For the values x that are close to 2, is this solution increasing or decreasing? Explain your answer.

SINCE WHEN $X=2$, $y=0$, SO $y'=0-2+1=-1$
SO NEAR $X=2$, FOR THIS SOLUTION,
$y(x)$ IS DECREASING.

(c) The pictures below show graphs of three functions; one of them is a solution of the equation $y' = xy - x + 1$, the other two are not. Which picture shows the solution? Explain carefully why the functions in the other two pictures **cannot** be solutions of the given equation.

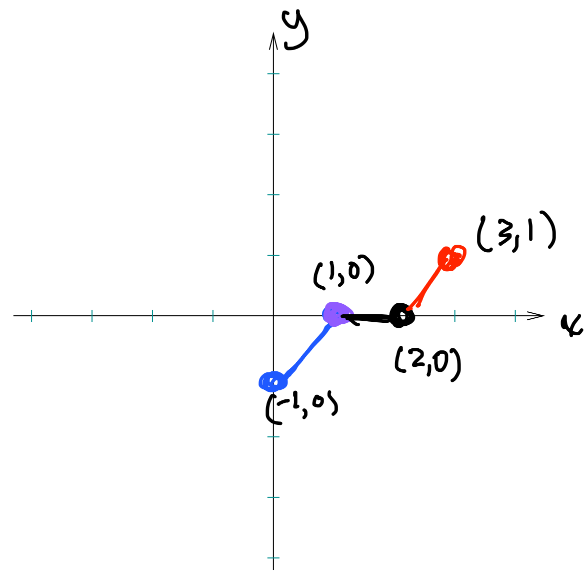


- A CANNOT BE A SOLUTION, BECAUSE IT HAS $y' \approx 0$ AT $(3,1)$. BUT $y' = xy - x + 1 = 1$ AT $(3,1)$.

- B CANNOT BE A SOLUTION SINCE AT ABOUT $(-2,0)$ $y' = +2+1 = 3$ BUT B HAS NEGATIVE SLOPE THERE.

C SEEMS TO BE OK.

(d) Use the Euler method with step size $h = 1$ to estimate $y(3)$, where $y(x)$ is the solution of initial value problem $y' = xy - x + 1$, $y(0) = -1$. The Euler method gives an approximation of the solution on the interval $[0, 3]$; sketch the graph of this approximation on the grid below. Please label the axes and all relevant points; show all work.



AT $x_0=0$, $y_0=-1$. $y_0' = 0-0+1 = 1$

$x_1=1$, $y_1 = y_0 + hy_0' = -1+1 = 0$. $y_1' = 0-1+1 = 0$

$x_2=2$, $y_2 = y_1 + hy_1' = 0+0=0$ $y_2' = 0-0+1 = 1$.

$x_3=3$, $y_3 = y_2 + hy_2' = 0+1 = 1$

7. This is a question about radioactive decay.

For your convenience, a few values and properties of logarithms are included at the bottom of the page (some will be useful, some are completely irrelevant). Properties of the exponential and log functions are also useful when you need to evaluate certain logarithms.

At 12:00pm, you get a sample of a rare radioactive element calculium-127. The mass of the sample is 100mg. One hour later, the mass of the sample is reduced by 80% (so that only 20% of the mass remains).

(a) Write the differential equation describing the decay of calculium-127; find all the necessary parameters.

$$\boxed{y' = ky, \quad y(0) = 100, \quad y(1) = 20}$$

$y(t) = Ae^{kt}$

$y(0) = 100 \Rightarrow A = 100$

$y(1) = 20 \Rightarrow 100e^{k} = 20$

$\Rightarrow e^{k} = \frac{1}{5}$

$k = \ln\left(\frac{1}{5}\right) = -\ln 5$

(b) Find a formula for the mass that remains after t hours.

$$y(t) = 100e^{-t\ln 5}$$

(c) At what time was the mass of the sample 25 times greater than its mass at 12:00pm?

WE NEED TO FIND t SO THAT

$y(t) = 2500$

$100e^{-t\ln 5} = 2500$

$e^{-t\ln 5} = 25$

$-t\ln 5 = \ln(25) = 2\ln 5$

$t = -2$

$\boxed{\text{SO, AT 10PM}}$

$\ln 5 \approx 1.6 \qquad \ln 20 \approx 3 \qquad \ln 0.8 \approx -0.22 \qquad \log_2 0.25 = -2$

$\ln(ab) = \ln a + \ln b \qquad e^{\ln a} = a$