Name: SOLUTIONS

ID#: ____________________

Test # 1

MAT 127 Spring 2005

Directions: There are 5 questions. You have until 10 PM (90 minutes). For credit, you must show all your work, using the backs of the pages if necessary. You may not use a calculator.

1. ____/20 2. ____/20 3. ____/20 4. ____/20 5. ____/20

Total Score. ____/100

1. A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 5y^3 + 6y^2.$$

(a) What are the constant solutions of the equation?

(b) For what values of y is y increasing?

(c) For what values of y is y decreasing?

(a) A SOLUTION IS CONSTANT IFF $\frac{dy}{dt} = 0$, SO WE SOLVE

$$0 = y^4 - 5y^3 + 6y^2$$

$$0 = y^2(y-2)(y-3)$$

HENCE THE CONSTANT SOL'NS ARE $\boxed{y=0,\ y=2,\ \text{AND } y=3}$

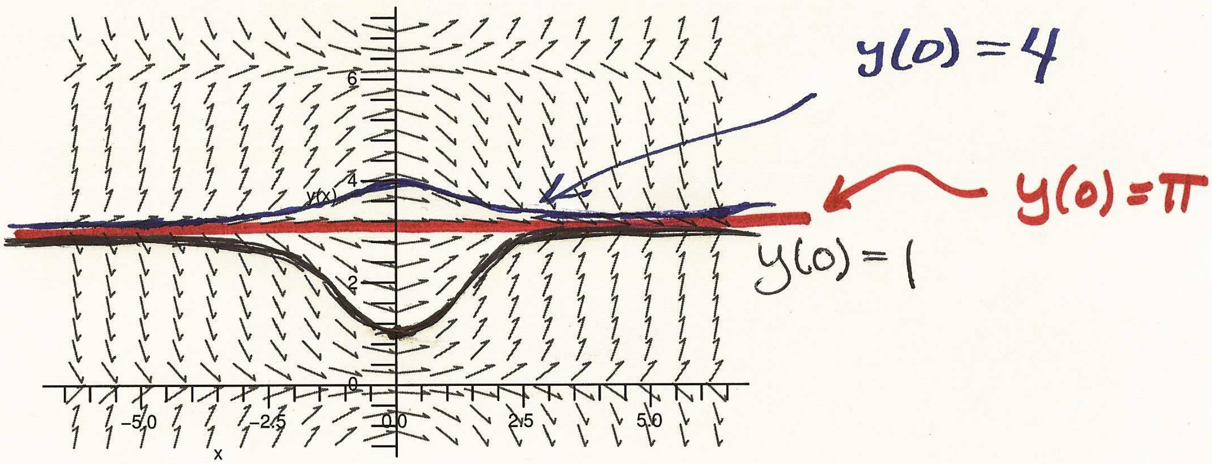
(b) THE SOL'N IS INCREASING WHEN $y' > 0$, IE, WHEN $y-2<0$ AND $y-3<0$ OR $y-2>0$ AND $y-3>0$.

IE, FOR $\boxed{y<2 \text{ OR } y>3}$

(c) DECREASING WHEN $\boxed{2<y<3}$

2. A direction field for the differential equation $y' = x \sin y$ is shown.

(a) Sketch the solutions that satisfy the given initial conditions (i) $y(0) = 1$, (ii) $y(0) = 4$,and (iii) $y(0) = \pi$. Label your graphs clearly.



(b) Find all equilibrium solutions.

WE'LL HAVE AN EQUILIBRIUM SOLUTION (IE, A CONSTANT SOL ~~≢~~ $y(x) = c$) WHEN $y' = 0$, IE $x \sin y = 0$.

THIS HAPPENS ONLY WHEN c IS SUCH THAT $\sin(c) = 0$, THAT IS ANY ~~#~~ EVEN MULTIPLE OF π.

SO, CONSTANT SOLUTIONS ARE

$$\boxed{y(x) = 2n\pi \quad \text{FOR EVERY INTEGER } n}$$

3. Use Euler's method with step size $1/2$ to estimate $y(2)$ for the solution to the initial value problem

$$y' = y - 2x \qquad y(1) = 0.$$

OUR INITIAL POINT IS

$$(x_0, y_0) = (1, 0)$$

GIVEN BY OUR INITIAL CONDITION.

THE NEXT APPROXIMATION WILL BE

$$\begin{cases} x_1 = x_0 + h \\ y_1 = y_0 + h \cdot y'(x_0, y_0) \end{cases}$$

IN OUR CASE, $y'(1,0) = 0 - 2 \cdot 1 = -2$ AND $h = \frac{1}{2}$

SO

$$(x_1, y_1) = \left(\tfrac{3}{2}, -1\right) \qquad \left(\text{ie, } y\left(\tfrac{3}{2}\right) \approx -1\right)$$

$$\begin{cases} x_2 = x_1 + h \\ y_2 = y_1 + h \cdot y'(x_1, y_1) \end{cases}$$

SO

$$x_2 = \frac{3}{2} + \frac{1}{2} = 2$$

$$y_2 = -1 + \frac{1}{2}\left(-1 - 2 \cdot \frac{3}{2}\right) = -1 - 2 = -3.$$

SO EULER'S w/ $h = \frac{1}{2}$ GIVES

$$\boxed{y(2) \approx -3}$$

4. Solve the following initial value problems. (Hint: the differential equations are seperable.)

(a) $\frac{dx}{dt} = 1 + x + t + tx$, $x(0) = 0$

(b) $\frac{dy}{dt} = 2te^y$, $y(1) = 0$

(a) $\frac{dx}{dt} = 1+x + t(1+x) = (1+t)(1+x)$

$\therefore \int \frac{dx}{1+x} = \int (1+t)dt$

$\ln|1+x| = \frac{1}{2}(1+t)^2 + C$

SINCE $x(0)=0$,

$\ln(1) = \frac{1}{2}(1)^2 + C$

$0 = \frac{1}{2} + C$, so $C = -\frac{1}{2}$

$\ln|1+x| = \frac{1}{2}(1+2t+t^2) - \frac{1}{2}$

$\ln|1+x| = t + t^2/2$

EXPONENTIATING,

$1+x = \pm e^{t+t^2/2}$ (WE WANT THE +, SINCE $x(0)=0$.)

SO $\boxed{x(t) = e^{t+\frac{t^2}{2}} - 1}$ (a)

(b) $\frac{dy}{dt} = 2te^y$

$\int \frac{dy}{e^y} = 2t\,dt$

$-e^{-y} = t^2 + C$

SINCE $y(1)=0$,

$-e^0 = -1 = 1 + C$

SO $C = -2$.

$\Rightarrow -e^{-y} = t^2 - 2$

$\Rightarrow e^{-y} = 2 - t^2$

$-y = \ln(2-t^2)$

SO

$\boxed{y(t) = -\ln(2-t^2)}$ (b)

5. Assume a population of well fed rabbits grows at a rate proportional to its size. Initially there are 100 rabbits and after 10 months there are 500 rabbits.

(a) Find an expression for the number of rabbits after t months.

(b) When will there be 5000 rabbits?

LET $P(t)$ = # OF RABBITS AFTER t MONTHS.

SO $P'(t) = kP(t)$

$\therefore P(t) = Ae^{kt}$ FOR SOME A & k.

SINCE $P(0) = 100$, $A = 100$

SINCE $P(10) = 500$, $500 = 100e^{10k}$, SO $5 = e^{10k}$, $\ln 5 = 10k$

SO $k = \frac{\ln 5}{10}$

$\boxed{(a)\ P(t) = 100e^{\frac{\ln 5}{10}t}}$

(b) MUST SOLVE $5000 = 100e^{\frac{\ln 5}{10}t}$ FOR t.

$50 = e^{\ln 5/10\, t}$

$\ln(50) = (\frac{1}{10}\ln 5)t$

SO $\boxed{t = \frac{10\ln 50}{\ln 5}}$

SINCE $\ln 50$

$= \ln(2 \cdot 5 \cdot 5)$

$= 2\ln 5 + \ln 2$

SO $\frac{10\ln 50}{\ln 5} = 20 + \frac{10\ln 2}{\ln 5}$

ABOUT 24 MONTHS