MAT 127 Solutions to First Midterm

1. A culture of bacteria grows at a rate proportional to the number of bacteria present in the culture. At noon on January 24, there were 15 thousand bacteria. At 2 PM, there were 60 thousand present.

   (a) 12 points Give a formula for $B(t)$, the number of bacteria in the culture t hours after noon on January 24.

   Solution: Since the growth rate of the bacteria is proportional to the number present, we have the differential equation

   $$B'(t) = kB(t)$$

   where t is the time in hours since noon, and $B(t)$ is the number of bacteria in thousands.

   This is a separable equation, so we separate variables to obtain

   $$\int \frac{dB}{B} = \int k\, dt$$

   $$\ln |B| = kt + c$$

   exponentiating both sides,

   $$|B| = e^{kt+c}$$

   so, since we can write $\pm e^c$ as an arbitrary constant A, we have

   $$B = Ae^{kt}.$$

   (Many students just remembered the formula for exponential growth and skipped directly to this step. That's fine, too.)

   From the initial condition, we know $B(0) = 15 = Ae^0 = A$. Since we also have $B(2) = 60$, we can solve for k:

   $$60 = 15e^{2k}$$

   $$4 = e^{2k}$$

   Taking logs,

   $$\ln 4 = 2k$$

   so $k = \frac{\ln 4}{2}$, or $k = \ln 2$.

   So

   $$B(t) = 15e^{t \ln 2} = 15 \cdot 2^t.$$

   (either form is OK. Many people also wrote $15e^{\frac{\ln 4}{2}t}$, which is equivalent.

(b) [8 points] When will there be 100 thousand bacteria in the culture?

Solution: To answer this, we need to find the value of t so that $B(t) = 100$. Since we have $B(t) = 15e^{t \ln 2}$ from the previous part, we solve

$$100 = 15e^{t \ln 2}$$

$$\frac{20}{3} = e^{t \ln 2}$$

Now take the log of both sides,

$$\ln \frac{20}{3} = t \ln 2$$

$$\frac{\ln (20/3)}{\ln 2} = t$$

That is, about 2.7 hours after noon.

2. [20 points] Consider the initial value problem given by

$$y' = x - 3y \qquad y(0) = 0$$

Use Euler's method with a stepsize $h = 1$ to find an approximation to $y(3)$.

To receive full credit, show your intermediate steps *clearly*.

Solution: Our initial point on our numeric solution is $(x_0, y_0) = (0, 0)$. The next approximation is given by $x_1 = x_0 + h$ and $y_1 = y_0 + h \cdot y'(x_0, y_0)$, so we need to find the slope of the solution at $(0, 0)$. Since our stepsize $h = 1$, things are easier.

$$y'(0,0) = 0 - 3 \cdot 0 = 0 \qquad \text{so} \quad (x_1, y_1) = (1, 0 + 0) = (1, 0).$$

Now we compute the slope at $(1, 0)$ for the next point. We have

$$y'(1,0) = 1 - 3 \cdot 0 = 1 \qquad \text{so} \quad (x_2, y_2) = (2, 0 + 1) = (2, 1).$$

Continuing in this way,

$$y'(2,1) = 2 - 3 \cdot 1 = -1 \qquad \text{so} \quad (x_3, y_3) = (3, 1 - 1) = (3, 0).$$

Our final approximation is then $y(3) = 0$.

3. Consider the second order linear differential equation

$$y'' - 9y = 0$$

(a) 10 points Write a formula for the general solution $y(t)$.

Solution: We look for solutions of the form $y = e^{kt}$, so we plug this in to get

$$k^2 e^{kt} - 4e^{kt} = 0.$$

This factors as

$$e^{kt}(k-2)(k+2) = 0,$$

which only has solutions when $k = 2$ or $k = -2$. This means the general solution to this differential equation is

$$y = Ae^{2t} + Be^{-2t},$$

where A and B are arbitrary constants.

(b) 10 points Let $y(t)$ be the specific solution with $y(0) = 1$ and $y'(0) = 0$. Write a formula for $y(t)$.

Solution: We need to determine A and B subject to the given initial conditions. From $y(0) = 1$, we have

$$1 = Ae^0 + Be^0 = A + B \qquad \text{so} \quad B = 1 - A.$$

That is, $y(t) = Ae^{2t} + (1-A)e^{-2t}$, and so

$$y'(t) = 2Ae^{2t} - 2(1-A)e^{-2t}$$

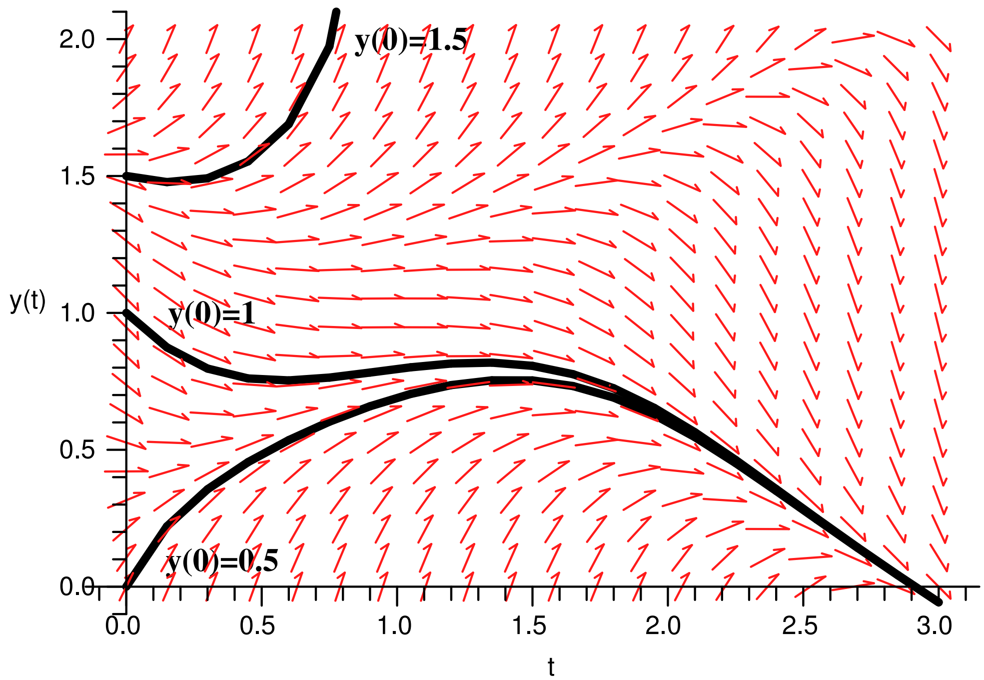
Plugging in $y'(0) = 0$ gives us

$$0 = 2A - 2(1-A) \qquad \text{that is} \quad 0 = 4A - 2 \quad \text{or} \quad A = \frac{1}{2}$$

Hence, $B = 1/2$ and our solution is

$$y(t) = \frac{e^{2t}}{2} + \frac{e^{-2t}}{2}$$

4. The direction field for a differential equation is shown below.



(a) 15 points On the direction field, sketch and **clearly label** the three solutions with initial conditions

$$y_1(0) = 0 \qquad y_2(0) = 1 \qquad y_3(0) = 1.5$$

(b) 5 points Are there any equilibrium solutions (also called stationary solutions, or constant solutions)? If there are, identify them. If not, give a reason why not.

Solution: There are no equilibrium solutions (at least not for $0 \le y \le 2$).
If there were, such a solution would be of the form $y(x) = c$ for some constant c, and its graph would be a horizontal line. Along this solution, the direction field must be slope 0 for all x. Since there are no such lines in the given direction field, we can have no equilibrium solutions.

5. Write solutions to the following initial-value problems.

   (a) 10 points $y' = \dfrac{e^{5x}}{y^4}$ $\qquad y(0) = -1$

Solution: This is a separable equation, so we separate the variables to obtain

$$\int y^4 \, dy = \int e^{5x} \, dx$$

and so

$$\frac{y^5}{5} = \frac{e^{5x}}{5} + c$$
$$y = \sqrt[5]{e^{5x} + c}$$

Now using the initial condition $y(0) = -1$, we have

$$-1 = \sqrt[5]{1 + c}$$

So $c = -2$ and our solution is

$$y = \sqrt[5]{e^{5x} - 2}$$

   (b) 10 points $y' = 1 + y^2$ $\qquad y(1) = 0$

Solution: This equation is also separable. Separating gives

$$\int \frac{dy}{1 + y^2} = \int dx$$

so

$$\arctan y = x + c \qquad \text{and hence} \qquad y = \tan(x + c)$$

The initial condition gives us $0 = \tan(1 + c)$, and since $\tan 0 = 0$, we know that $c = -1$. Hence the desired solution is

$$y = \tan(x - 1)$$