Math 126 Solutions to First Midterm, emu version

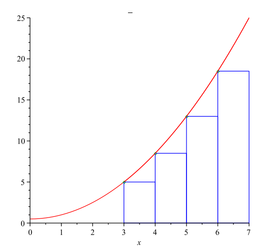
1. Let
$$f(x) = \frac{1}{2}x^2 + \frac{1}{2}, \qquad 3 \le x \le 7.$$

(In what follows you are not asked to compute the exact value of a definite integral.)

$\boxed{\text{20 points}}$ (a) Use the left endpoint rule and four intervals to estimate the area of the region under the graph of the function.

Solution: Since we are dividing the interval $[3, 7]$ into 4 equal segments, we have $\Delta x = \frac{7-3}{4} = 1$. Thus, we take $x_1 = 3$, $x_2 = 4$, $x_3 = 5$, and $x_4 = 6$.
Our approximation is then

$$1 \cdot \Big(f(3) + f(4) + f(5) + f(6)\Big) = \frac{9}{2} + \frac{17}{2} + \frac{26}{2} + \frac{37}{2} = \frac{89}{2}$$



$\boxed{\text{10 points}}$ (b) Is the estimate you have obtained an overestimate or an underestimate?

Solution: Since $f(x)$ is increasing, the left estimate is an underestimate.

$\boxed{\text{10 points}}$ (c) If you apply the right endpoint rule to the same function, interval and number of intervals, do you obtain an overestimate or an underestimate?

Solution: Since $f(x)$ is increasing, the right estimate would be an overestimate.

2. Consider the limit of the sum
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \left(\frac{4}{n} i\right)^3. \tag{1}$$

$\boxed{\text{20 points}}$ (a) Find a function $f(x)$ and an interval $[a, b]$ such that the limit above is the area under the graph of $f(x)$ over the interval $[a, b]$.

Solution: Recall the definition of the integral (using right sums) is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x) f\left(a + i\Delta x\right)$$

Here we can take $\Delta x = 4/n$, and we can take $a = 0$, so that $b = 4$, and $f(x) = x^3$.

Thus, we have the sum equivalent to $\displaystyle\int_0^4 x^3\,dx$.

20 points

(b) Use the following formula for the sum of the first n cubes

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

to compute the limit of the sum in (1) (do not compute the integral directly).

Solution: Before computing the limit, let's do some algebra to simplify things a bit first. We have

$$\frac{4}{n}\left(\frac{4}{n}i\right)^3 = \frac{4^4}{n^4}i^3,$$

and so

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left(\frac{4}{n}i\right)^3 = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4^4}{n^4}i^3 = \lim_{n\to\infty}\frac{4^4}{n^4}\sum_{i=1}^{n}i^3 = \lim_{n\to\infty}\frac{4^4}{n^4}\cdot\frac{n^2(n+1)^2}{4}.$$

Since $\lim_{n\to\infty}\frac{n^2(n+1)^2}{n^4} = 1$, we have shown that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\left(\frac{4}{n}i\right)^3 = \frac{4^4}{4} = 64.$$

Although the question says not to compute the integral directly, it is a good idea to check the answer by doing so. We have

$$\int_0^4 x^3\,dx = \left.\frac{x^4}{4}\right|_0^4 = \frac{4^4}{4} - 0 = 64,$$

which agrees with the answer from the direct computation of the limit. ka-POW!

3.

20 points

(a) A function $f(x)$ has maximum value -2 and minimum value -5 on the interval $[-3,3]$. Between what two values must

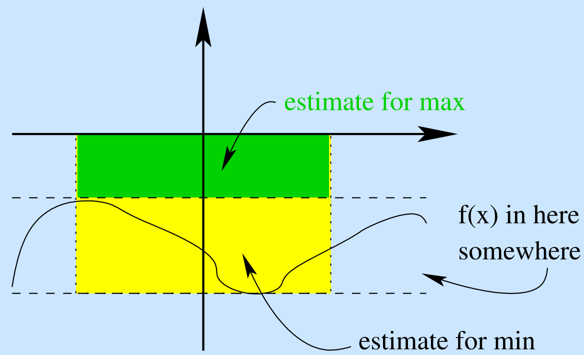
$$\int_{-2}^{2} f(x)\,dx$$

lie?

Solution: Many people were confused by this, but it is really very simple if you just read the problem carefully. It says that for $-3 \le x \le 3$, we have

$-5 \leq f(x) \leq -2$. We are asked to find two numbers between which $\int_{-2}^{2} f(x)\,dx$ must lie.

Since we don't know anything about $f(x)$ other than its maximum and minumum, we can use these to calculate an upper and a lower approximation for the integral using just one rectangle. This is the same as the statement on one of the homework problems that



$$\text{If} \quad m \leq f(x) \leq M, \qquad \text{then} \qquad (b-a)m \leq \int_b^a f(x)\,dx \leq (b-a)M.$$

Since $a = -2$ and $b = 2$, the width is 4. Since $f(x) < -2$, we can take the height of the upper (green) rectangle to be -2, giving us an estimate of -8. For the lower (yellow) rectangle, we have $-5 \leq f(x)$, so we have an estimate of -20. Thus, we can conclude that

$$-20 \leq \int_{-2}^{2} f(x)\,dx \leq -8.$$

20 points

(b) Without computing the actual value, use the properties of definite integrals to estimate from above and from below the integral

$$\int_{-1}^{2} 1 + 4e^{x^2}\,dx.$$

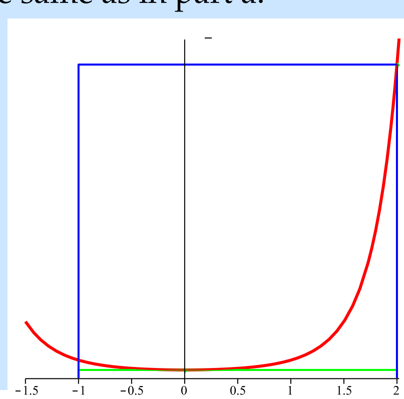
You must give a correct justification of your answer to receive credit.

Solution: The easiest way to do this problem is the same as in part a.

Observe that for $-1 \leq x \leq 2$, we know that $5 \leq 1 + 4e^{x^2} \leq 1 + 4e^4$. (Note that the minimum is at $x = 0$, not at $x = -1$).

Applying the same idea as the first part, we can fit a rectangle of width 3 and height 4 under the curve, and one of width 3 and height $1 + 4e^4$ over it. Thus, we can conclude that

$$15 \leq \int_{-1}^{2} 1 + 4e^{x^2}\,dx \leq 3(1 + 4e^4).$$



If, instead you chose to observe that $1 + 4e^{x^2}$ is always positive and is less than 1000 for $|x| < 2$, you could just as well have given 0 for your lower bound and 3000 for the upper.

You can *not* do this problem by calculating the antiderivative directly, because e^{x^2} does not have an antiderivative that can be written in terms of elementary functions.

Quite a few people found their estimates using a right sum and a left sum, and claimed that the right sum was an overestimate and the left sum was an underestimate. This is not quite correct, since the function is decreasing when $x < 0$ and increasing for $x > 0$. This makes it much harder to tell whether the right sum is an underestimate or an overestimate.

4. Evaluate the definite integrals in parts (a) and (b). Determine the general indefinite integrals in parts (c) and (d).

$\boxed{\text{10 points}}$ (a) $\displaystyle\int_{-1}^{1} \frac{4}{t^2+1}\,dt$

Solution:

$$\int_{-1}^{1} \frac{4}{t^2+1}\,dt = 4\int_{-1}^{1} \frac{1}{t^2+1}\,dt = 4\Big(\arctan(1) - \arctan(-1)\Big) = 4\Big(\frac{\pi}{4} + \frac{\pi}{4}\Big) = 2\pi.$$

$\boxed{\text{10 points}}$ (b) $\displaystyle\int_{0}^{3} |2x-1|\,dx$

Solution: Observe that if $x < 1/2$, we have $|2x-1| = -(2x-1) = 1-2x$ but for $x > 1/2$, we have $|2x-1| = 2x-1$. This means that we can calculate the integral as follows:

$$\begin{aligned}\int_0^3 |2x-1|\,dx &= \int_0^{1/2} 1-2x\,dx + \int_{1/2}^3 2x-1\,dx \\ &= (x-x^2)\Big|_0^{1/2} + (x^2-x)\Big|_{1/2}^3 \\ &= \Big(\frac{1}{2}-\frac{1}{4}\Big) + \Big((9-3) - \big(\frac{1}{4}-\frac{1}{2}\big)\Big) = \frac{1}{4}+6+\frac{1}{4} = \frac{13}{2}\end{aligned}$$

10 points (c) $\int \sec w \tan w \, dw$

Solution:

$$\int \sec w \tan w \, dw = \sec w + C$$

10 points (d) $\int (e^x + \frac{1}{x} - \frac{3}{x^3}) \, dx$

Solution:

$$\int (e^x + \frac{1}{x} - \frac{3}{x^3}) \, dx = e^x + \ln|x| + \frac{3}{2x^2} + C$$

5. A particle is moving along a line [1]. Its velocity $v(t)$ at time 0 is equal to: $v(0) = 0$. Its acceleration is known and equal to:

$$a(t) = 3t - 6, \qquad 0 \le t \le 4.$$

20 points (a) Find the velocity $v(t)$ in the given interval of time.

Solution: Recall that the derivative of velocity is acceleration. Thus,

$$v(t) = \int 3t - 6 = \frac{3}{2}t^2 - 6t + C,$$

Since we know $v(0) = 0$, this tells us that $C = 0$, and so

$$v(t) = \frac{3}{2}t^2 - 6t.$$

20 points (b) The position $s(t)$ of the particle at time 0 is equal to: $s(0) = 0$. Find its position at time $t = 4$.

Solution: Using the previous part, we have $v(t) = 3t^2/2 - 6t$. Since $s'(t) = v(t)$, and since we want $s(0) = 0$, the FTC tells us that

$$s(t) = \int_0^t \frac{3x^2}{2} - 6x \, dx = \frac{x^3}{2} - 3x^2.$$

Thus

$$s(4) = \frac{64}{2} - 48 = -16$$

[1] It does not matter for your answer, but in case you wonder: distance is measured in meters and time is measured in seconds.

6. Use the fundamental theorem of calculus (FTC) to answer the following questions.

15 points

(a) Find the derivative $g'(x)$ of the function $g(x) = \int_x^1 \sin(t^2+1)\,dt$.

Solution: Since

$$g(x) = \int_x^1 \sin(t^2+1)\,dt = -\int_1^x \sin(t^2+1)\,dt,$$

we can apply the fundamental theorem directly to conclude that

$$g'(x) = -\sin(x^2+1)$$

15 points

(b) Find the derivative $h'(x)$ of the function $h(x) = \int_{\sin x}^2 e^{4t}\,dt$.

Solution: Here we have to pay a bit more attention and use the chain rule, because we are not taking the derivative with respect to the variable in the limits of integration. Thus, we have

$$\int_{\sin x}^2 e^{4t}\,dt = e^8\left(\frac{d}{dx}2\right) - e^{4\sin x}\left(\frac{d}{dx}\sin x\right) = 0 - e^{4\sin x}(\cos x) = -\cos x\, e^{4\sin x}.$$

20 points

(c) Use the FTC to define the function $f(x)$ with these properties: $f'(x) = 2^x e^{\sin x}$ and $f(3) = 3$.

Solution: First, observe that for any choice of a, the Fundamental Theorem of Calculus tells us that $f(x) = \int_a^x 2^t e^{\sin t}$ satisfies $f'(x) = 2^x e^{\sin x}$; furthermore, for such an f we have $f(a) = 0$.
Once we have that, it is easy to come up with the desired function: set $a = 3$, and just add 3 to the result. Thus, the following function has the desired properties:

$$f(x) = 3 + \int_3^x 2^t e^{\sin t}\,dt$$

Despite what many students claimed, it is *not possible* to find a nice formula for the antiderivative of $2^x e^{\sin x}$.