MATH 126 Solutions to Midterm 2, Fall 2015

10 pts 1. $\displaystyle\int \sin^2(4x)\,dx$

Solution: Using the identity $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$ (with $\theta = 2x$), we get

$$\int \sin^2(4x)\,dx = \tfrac{1}{2}\int 1-\cos(8x)\,dx = \tfrac{1}{2}x - \tfrac{1}{2}\int \cos(8x)\,dx = \boxed{\frac{x}{2} - \frac{\sin(8x)}{16} + C}$$

where we used the substitution $u = 8x \quad du = 8dx \quad \frac{du}{8} = dx$ in the second integral.

10 pts 2. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x\,dx$

Solution: Here we use the identity $\cos^2 x = 1 - \sin^2 x$ so that we can make the subsitution $u = \sin x \quad du = \cos x\,dx$ and have a lovely time. When we do that, observe that if $x = 0$, $u = \sin(0) = 0$ and when $x = \pi/2$, $u = \sin(\pi/2) = 1$.

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \sin^5 x(1-\sin^2 x)\cos x\,dx = \int_0^1 u^5(1-u^2)\,du = \frac{u^6}{6} - \frac{u^8}{8}\bigg|_0^1 = \frac{1}{6} - \frac{1}{8} = \boxed{\frac{1}{24}}.$$

10 pts 3. $\displaystyle\int \frac{8x-7}{x^2-x-2}\,dx$

Solution: Observe that $x^2 - x - 2 = (x-2)(x+1)$, so partial fractions will be helpful here. So we want to find A and B so that

$$\frac{8x-7}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \quad \text{or} \quad 8x - 7 = A(x+1) + B(x-2).$$

Since this must be true for every x, we can choose helpful values of x to make things easier.

If $x = 2$, we get $\quad 16 - 7 = A \cdot 0 + B \cdot 3, \quad$ so $\quad B = 9/3 = 3.$

When $x = -1$, $\quad -8 - 7 = A \cdot (-3) + B \cdot 0, \quad$ so $\quad A = 15/3 = 5.$

If you prefer to multiply things out, match coefficients, and solve the resulting equations, you would get $8x - 7 = Ax - 2A + Bx + B$, which means you need to solve

$$8 = A + B \qquad -7 = -2A + B.$$

Subtracting the second from the first gives $15 = 3A$, or $A = 3$. Substituting that into the first equation gives $8 = 3 + B$, so $B = 5$.

If you don't make a mistake, you always get the same answer, so either method is fine.

Using $A = 5$, $B = 3$ gives us

$$\int \frac{8x-7}{x^2-x-2}\,dx = \int \frac{3\,dx}{x-2} + \int \frac{5\,dx}{x+1} = \boxed{3\ln|x-2| + 5\ln|x+1| + C}.$$

10 pts 4. $\displaystyle\int_2^\infty \frac{dt}{t^3}$

Solution: This is an improper integral, so we must compute it as a limit.

$$\int_2^\infty \frac{dt}{t^3} = \lim_{M\to\infty}\int_2^M \frac{dt}{t^3} = \lim_{M\to\infty}\left(-\frac{1}{2t^2}\bigg|_2^M\right) = \lim_{M\to\infty}\left(-\frac{1}{2M^2}+\frac{1}{8}\right) = 0 + \frac{1}{8} = \boxed{\frac{1}{8}}.$$

15 pts 5. $\displaystyle\int \sec^4 x\,dx$

Solution: Since the derivative of $\tan x$ is $\sec^2 x$ and we know that $\sec^2 x = 1 + \tan^2 x$, we can write $\sec^4 x = (1+\tan^2 x)\sec^2 x$. This will be helpful, since then we can make the substitution $u = \tan x \;\; du = \sec^2 x\,dx$, and then party like it's 1999 (though, sadly, without Prince).

$$\int \sec^4 x\,dx = \int (1+\tan^2 x)\sec^2 x\,dx = \int (1+u^2)\,du = u + \frac{u^3}{3} + C = \boxed{\tan x + \frac{\tan^3 x}{3} + C}.$$

15 pts 6. $\displaystyle\int_4^{12} \frac{dx}{\sqrt[3]{x-4}}$

Solution: Here we make the substitution $w = x - 4$, so $dw = dx$; when $x = 4$, $w = 0$ and when $x = 12$, $w = 8$. The resulting integral is improper at $w = 0$ (or $x = 4$), so we need to write it as a limit.

$$\int_4^{12} \frac{dx}{\sqrt[3]{x-4}} = \int_0^8 \frac{dx}{\sqrt[3]{w}} = \lim_{t\to 0^+}\int_t^8 \frac{dx}{\sqrt[3]{w}} = \lim_{t\to 0^+}\left(-\tfrac{3}{2}w^{2/3}\bigg|_t^8\right) = \lim_{t\to 0^+}\left(-\tfrac{3}{2}t^{2/3}+6\right) = 0 + 6 = \boxed{6}.$$

Remember that $\frac{3}{2}\cdot 8^{2/3} = \frac{3}{2}(\sqrt[3]{8})^2 = 12/2 = 6$.

15 pts 7. $\int \frac{x^2 - x - 6}{(x^2+1)(2x-1)}\, dx$

Solution: This is another partial fractions problem. We write

$$\frac{x^2 - x - 6}{(x^2+1)(2x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{2x-1},$$

remembering that the numerator in each term must be a polynomial one degree lower than the denominator. This means we need to find A, B, and C so that

$$x^2 - x - 6 = (Ax+B)(2x-1) + C(x^2+1).$$

Setting $x = \frac{1}{2}$ gives $\frac{1}{4} - \frac{1}{2} - 6 = 0 + C(\frac{1}{4}+1)$, that is, $-\frac{25}{4} = \frac{5}{4}C$ or $C = -5$.

Now we set $x = 0$ to get $-6 = -B + C$. But since $C = -5$, we obtain $B = 1$.

Finally, we pick some other value of x, for example, $x = 1$. This gives us $1 + 1 - 6 = (A+B)(2+1) + C(1+1)$ or $-6 = 3A + 3B + 2C$. But we already know $B = 1$ and $C = -5$, so we have $-4 = 3A + 3 - 10 = 3A - 7$. That is, $3 = 3A$ or $A = 1$.

If you prefer the other way, multiplying everything out and collecting terms gives you

$x^2 - x - 6 = (2A+C)x^2 + (-A+2B)x + (-B+C)$ or $\{2A+C = 1,\ -A+2B = -1,\ -B+C = -6\}$

These give the same solutions $A = 3$, $B = 1$, and $C = -5$, but the details are rather tedious, so I'm skipping them.

Now back the the calculus.

$$\begin{aligned}\int \frac{x^2 - x - 6}{(x^2+1)(2x-1)}\, dx &= \int \frac{3x\, dx}{x^2+1} + \int \frac{dx}{x^2+1} - 5\int \frac{dx}{2x-1} \\ &= \frac{3}{2}\int \frac{du}{u} + \arctan x - \frac{5}{2}\int \frac{dw}{w} \\ &= \boxed{\frac{3}{2}\ln(x^2+1) + \arctan x - \frac{5}{2}\ln|2x-1| + K},\end{aligned}$$

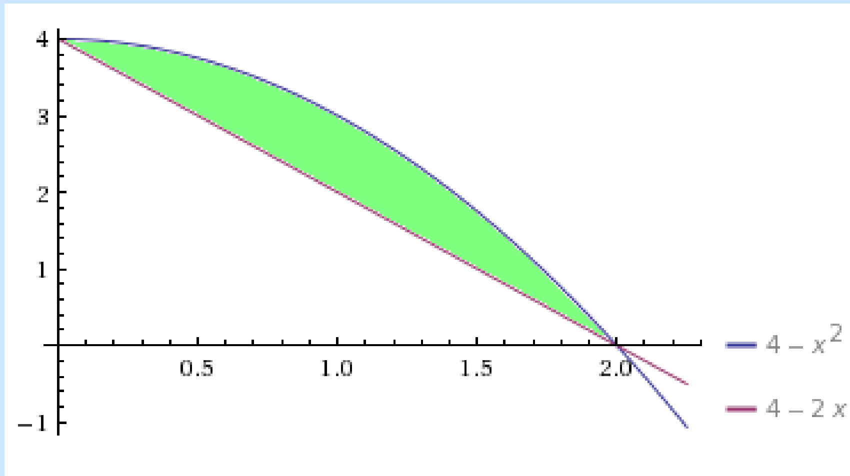
where we made the substitutions $u = x^2 - 1$ (so $du = 2x\, dx$) in the first integral and $w = 2x - 1$ (so $dw = 2dx$) in the third one.

8. The region R in the first quadrant is bounded by $y = 4 - x^2$ and $y = 4 - 2x$.

3 pts

(a) Sketch the region R.

Solution:



12 pts

(b) Find the area of R.

Solution: Note that the two curves cross at $x = 0$ and $x = 2$: we solve $4 - x^2 = 4 - 2x$, so we have $0 = x^2 - 2x = x(2 - x)$. Also notice that for $0 \leq x \leq 2$, $4 - x^2$ is larger than $4 - 2x$. So, the area is given by the integral

$$\int_0^2 (4 - x^2) - (4 - 2x)\,dx = \int_0^2 2x - x^2\,dx = \boxed{\frac{4}{3}}.$$