MAT 126 Calculus B Spring 2007
Practice Midterm I — Solutions

Name:____

I.D.:____________________________ **Section number:** __________

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. (a) (10 points) Estimate the area under the graph of $f(x) = 16 - x^2$ from $x = 0$ to $x = 4$ using four rectangles and right endpoints. Sketch the graph and rectangles. Is your estimate and underestimate or an overestimate?

Solution.

$$R_4 = f(1) + f(2) + f(3) + f(4) = 15 + 12 + 7 = 34$$

The function $f(x)$ is concave downward, so that $R_4 < A$, i.e., it is an underestimate (sketch the graph!)

(b) (10 points) Repeat part (a) using left endpoints.

Solution.

$$L_4 = f(0) + f(1) + f(2) + f(3) = 50$$

It is an overestimate, $L_4 > A$ (sketch the graph!)

2. (a) (10 points) Evaluate integral by interpreting it as area

$$\int_{-5}^{5} \sqrt{25 - x^2}dx$$

Solution. It is the upper half of the circle of radius 5 centered at the origin, so

$$\int_{-5}^{5} \sqrt{25 - x^2}dx = \frac{1}{2}\pi\, 5^2 = 12.5\pi$$

(b) (5 points) Determine a region whose area is equal to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$$

Do not evaluate the limit.

Solution. We identify $\Delta x = \pi/4n$. Since in general

$$\Delta x = \frac{b-a}{n},$$

we conclude that $b - a = \pi/4$. Comparing the general expression $f(x_i) = f(a + i\Delta x)$ with $\tan\frac{i\pi}{4n}$, we conclude that $f(x) = \tan x$ and $a = 0$. Thus we have the right endpoint sum for the function $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$, and

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n} = \int_0^{\pi/4} \tan x dx.$$

Thus the region is the region under the graph of $y = \tan x$ from $x = 0$ to $x = \pi/4$.

3. Given two functions $f(x)$ and $g(x)$ which satisfy

$$\int_0^3 f(x)dx = 5, \quad \int_0^5 f(x)dx = 7,$$
$$\int_3^5 g(x)dx = 1, \quad \int_0^5 g(x)dx = 9,$$

find

(a) (5 points)

$$\int_3^5 (3f(x) - g(x))dx$$

Solution. We have

$$\int_3^5 f(x)dx = \int_0^5 f(x)dx - \int_0^3 f(x)dx = 7 - 5 = 2$$

so that

$$\int_3^5 (3f(x) - g(x))dx = 3\int_3^5 f(x)dx - \int_3^5 g(x)dx = 6 - 1 = 5$$

(b) (5 points)

$$\int_0^3 (f(x) + 2g(x))dx$$

Solution. Similarly,

$$\int_0^3 g(x)dx = 9 - 1 = 8$$

and

$$\int_0^3 (f(x) + 2g(x))dx = 5 + 2 \times 8 = 21$$

4. (5 points) Express the limit as a definite integral on the given interval $[0, 4]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i} \Delta x$$

Solution.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i} \Delta x = \int_0^4 \frac{e^x}{1+x} dx$$

5. Evaluate the following indefinite integrals

(a) (5 points)

$$\int (3\cos x - 4\sin x)dx$$

Solution.

$$\int (3\cos x - 4\sin x)dx = 3\sin x + 4\cos x + C$$

(b) (10 points)

$$\int \frac{\cos x}{1 - \cos^2 x} dx$$

Solution. Using the fundamental trigonometric identity,

$$\int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C$$

6. Evaluate the following definite integrals

(a) (5 points)

$$\int_1^2 x^{-2} dx$$

Solution.

$$\int_1^2 x^{-2} dx = \left. \frac{x^{-1}}{-1} \right|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

(b)

$$\int_1^8 \frac{x - 1}{\sqrt[3]{x^2}} dx$$

Solution.

$$\int_1^8 \frac{x-1}{\sqrt[3]{x^2}}dx = \int_1^8 (x^{1/3} - x^{-2/3})dx$$
$$= \left(\frac{3}{4}x^{4/3} - 3x^{1/3}\right)\Big|_1^8 = \left(\frac{3}{4}\times 8^{4/3} - 3\times 8^{1/3}\right) - \left(\frac{3}{4} - 3\right) = 8.25$$

where we have used that $8 = 2^3$.

(c) (5 points)

$$\int_1^{27} \frac{1}{9t}dt$$

Solution.

$$\int_1^{27} \frac{1}{9t}dt = \frac{1}{9}\ln t\Big|_1^{27} = \frac{1}{9}\ln 27 = \frac{1}{3}\ln 3,$$

where we have used that $\ln 1 = 0$ and $27 = 3^3$.

(d) (5 points)

$$\int_{\ln 3}^{\ln 6} 5e^x dx$$

Solution.

$$\int_{\ln 3}^{\ln 6} 5e^x dx = 5e^x|_{\ln 3}^{\ln 6} = 5(e^{\ln 6} - e^{\ln 3}) = 5(6-3) = 15$$

(e) (10 points)

$$\int_{\pi/3}^{\pi/2} \csc x \cot x dx$$

Solution.

$$\int_{\pi/3}^{\pi/2} \csc x \cot x dx = \int_{\pi/3}^{\pi/2} \frac{\cos x}{\sin^2 x}dx$$
$$= \left(-\frac{1}{\sin x}\right)\Big|_{\pi/3}^{\pi/2} = -1 + \frac{2}{\sqrt{3}}.$$