Name: SOLUTIONS Id: SPR 2010

20 pts 1. (a) Express the following limit of Riemann sums as a definite integral. (do not compute the integral)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{x_i^2+e^{x_i}}\,\Delta x, \qquad \text{where } x_i = 2+i\Delta\, x, \quad \Delta\, x = \frac{1}{n}$$

WITH $\Delta x = \frac{1}{n}$ AND $x_i = 2 + i/n$, WE GET $2 \le x \le 3$

(TAKING $i=0$ TO GET 2, $i=n$ TO GET 3)

SO THE INTEGRAL IS

$$\boxed{\int_2^3 \sqrt{x^2+e^x}\,dx}$$

(OTHER VARIATIONS LIKE $\int_0^1 \sqrt{(2+x)^2+e^{2+x}}\,dx$ ARE CORRECT, BUT UNEXPECTED)

20 pts (b) Express the following integral as a limit of Riemann sums. (do not compute the integral)

$$\int_1^3 \sqrt{1+x^3}\,dx$$

SINCE $b-a = 3-1 = 2$, IT IS REASONABLE TO USE $\Delta x = \frac{2}{n}$ AND SO $x_i = 1 + \frac{2}{n}$, $f(x) = \sqrt{1+x^3}$

THEN WE HAVE

$$\int_1^3 \sqrt{1+x^3}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\cdot f(x_i) = \boxed{\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{1+\left(1+\frac{2}{n}\right)^3}}$$

(OTHER VARIATIONS ARE POSSIBLE)

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30 pts 2. Define a function $f(x) = \int_0^{x^2} \sqrt{t + \sqrt{t}}\, dt.$

Find the value of $f'(1)$.

WRITE $g(u) = \int_0^u \sqrt{t + \sqrt{t}}\, dt.$

THEN BY THE FUNDAMENTAL THEOREM OF CALCULUS,

$$g'(u) = \sqrt{u + \sqrt{u}}$$

SO, $f(x) = g(x^2) = \int_0^{x^2} \sqrt{t + \sqrt{t}}\, dt$

AND BY THE CHAIN RULE,

$$f'(x) = g'(x^2) \cdot 2x = \left(\sqrt{x^2 + \sqrt{x^2}}\right) \cdot 2x = 2x\sqrt{x^2 + |x|}$$

THUS

$$f'(1) = 2 \cdot 1 \cdot \sqrt{1+1} = \boxed{2\sqrt{2}}$$

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3. Determine whether each integral is convergent or divergent and evaluate those that are convergent (if any).

20 pts (a) $\displaystyle\int_{-\infty}^{0} \frac{1}{(-1-x)^{\frac{1}{3}}}\,dx = \int_{-\infty}^{0} -(1+x)^{-1/3}\,dx$

NOTE THAT THE INTEGRAND IS NOT DEFINED AT $X=-1$. WE NEED TO SPLIT THIS INTO THREE IMPROPER INTEGRALS:

ANY NUMBER <-1 WORKS

$$\int_{-\infty}^{-9} -(1+x)^{-1/3}\,dx + \int_{-9}^{-1} -(1+x)^{-1/3}\,dx + \int_{-1}^{0} -(1+x)^{-1/3}\,dx$$

$$= \lim_{m\to-\infty} \int_{M}^{-9} -(1+x)^{-1/3}\,dx + \lim_{t\to-1^-} \int_{-9}^{t} -(1+x)^{-1/3}\,dx + \lim_{r\to-1^+} \int_{r}^{0} -(1+x)^{-1/3}\,dx$$

$$= \lim_{m\to-\infty} \left(-\frac{3}{2}(1+x)^{2/3}\Big|_{M}^{-9}\right) + \lim_{t\to-1^-} \left(+\frac{3}{2}(-8)^{2/3} - \frac{3}{2}t^{2/3}\right) + \lim_{r\to-1^+} \left(+\frac{3}{2}r^{2/3} + 0\right)$$

$$= \left(-\frac{3}{2}\cdot 4 + \lim_{m\to-\infty} \frac{3}{2}(1+M)^{2/3}\right) + \frac{3}{2}\cdot 4 = \infty$$

INTEGRAL $\boxed{\text{DIVERGES!}}$

20 pts (b) $\displaystyle\int_{0}^{1} \frac{\ln x}{x}\,dx$

NOTE THAT $\displaystyle\int \frac{\ln x}{x}\,dx = \int u\,du = u^2/2 + C = \frac{(\ln x)^2}{2} + C$.

$u = \ln x$

$du = \dfrac{dx}{x}$

BUT $\dfrac{\ln x}{x}$ IS NOT DEFINED AT $X=0$, SO

$$\int_{0}^{1} \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \int_{t}^{1} \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \frac{(\ln x)^2}{2}\Big|_{t}^{1} = 0 - \lim_{t\to 0^+} \frac{(\ln t)^2}{2}$$

$$= \boxed{-\infty}$$

SO INTEGRAL $\boxed{\text{DIVERGES}}$

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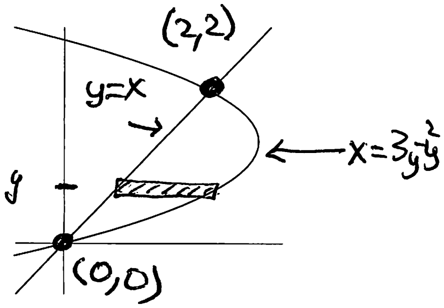
30 pts 4. Find the area of the region bounded by the two curves

$$x = 3y - y^2 \qquad \text{and} \qquad y = x.$$



(a) Write an integral which represents this area.

THE CURVES CROSS IF $3y - y^2 = y$

ie $y^2 - 2y = 0$

$y(y-2) = 0$, SO $y = 0$ OR $y = 2$

$x = 0$ $\quad x = 2$.

EASIEST TO INTEGRATE WRT y

A TYPICAL RECTANGLE IS $(3y - y^2) - y$ BY dy

SO THE AREA IS GIVEN BY

$$\boxed{\int_0^2 (2y - y^2)\,dy}$$

(b) Evaluate the integral in (a).

$$\int_0^2 (2y - y^2)\,dy = y^2 - y^3/3 \Big|_0^2$$

$$= \left(4 - \frac{8}{3}\right) - 0$$

$$= \frac{12}{3} - \frac{8}{3} = \boxed{\frac{4}{3}}$$

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30 pts 5. Compute the following integral. If the integral diverges, write "divergent".

$$\int_{-1}^{0} \frac{x}{(x-1)(x-2)}\,dx.$$

NOTE THAT $\frac{x}{(x-1)(x-2)}$ IS DEFINED FOR $x<1$, SO NOT IMPROPER.

USING PARTIAL FRACTIONS,

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

OR $x = A(x-2) + B(x-1)$

IF $x=1$, THEN $1 = -A + 0$, SO $A=-1$

IF $x=2$, THEN $2 = 0 + B$, SO $B=2$

$\boxed{\text{OR}}$

$x = Ax - 2A + Bx - B$

SO

$A+B=1$

$-2A-B=0$

ADDING GIVES

$-A=1$, SO $B=2$.

THUS

$$\int_{-1}^{0} \frac{x}{(x-1)(x-2)}\,dx = \int_{-1}^{0} \frac{-1}{x-1} + \frac{2}{x-2}\,dx$$

$$= -\ln|x-1| + 2\ln|x-2| \Big|_{-1}^{0}$$

$$= \left(-\ln(1) + 2\ln 2\right) - \left(-\ln(2) + 2\ln(3)\right)$$

$$= 3\ln 2 - 2\ln 3$$

$$= \ln 8 - \ln 9 = \ln\left(\tfrac{8}{9}\right)$$

ANY OF THESE ARE OK.

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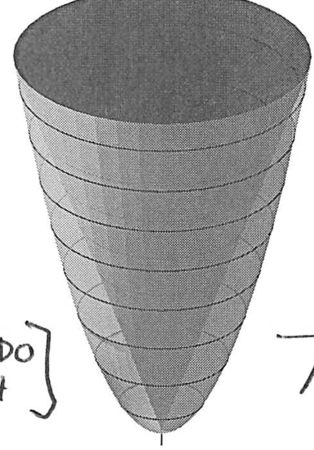
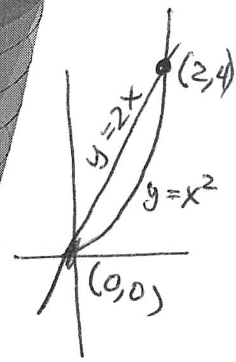
| 30 pts | 6. Find the volume of the solid obtained by rotating the region between the two curves

$$y = 2x \qquad \text{and} \qquad y = x^2$$

about the y-axis.

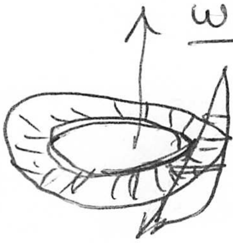




(a) Write an integral which represents the volume.

THE CURVES CROSS WHEN $2x = x^2$, THAT IS FOR $x(x-2)=0$, AT $x=0$, $x=2$
$y=0$, $y=4$.

WE CAN INTEGRATE BY WASHERS (dy) OR CYLINDERS (dx) [CHOOSE ONE, I'LL DO BOTH]

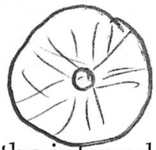


WASHERS (dy)

OUTERCURVE IS $x = \sqrt{y}$

INNER IS $x = y/2$

A SLICE IS A RING/WASHER WITH AREA

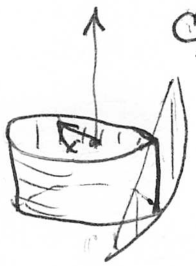


$$\underbrace{\pi(\sqrt{y})^2}_{\text{OUTER}} - \underbrace{\pi\left(y/2\right)^2}_{\text{INNER}},$$

INTEGRAL IS

$$\boxed{\int_0^4 \pi\left(y - y^2/4\right)dy}$$

~~(b) Evaluate the integral in (a).~~



CYLINDERS (dx)

SLICES ARE VERTICAL,

TOP IS $y = 2x$
BOTTOM IS $y = x^2$ } HEIGHT IS $2x - x^2$

CIRCUMFERENCE IS $2\pi x$

AREA IS $2\pi x(2x - x^2)$

INTEGRAL IS

$$\boxed{\int_0^2 2\pi\left(2x^2 - x^3\right)dx}$$

(b) EVALUATE

FOR WASHERS,

$$\pi\int_0^4 y - y^2/4 \, dx = \pi\left(y^2/2 - y^3/12\right)\Big|_0^4 = \pi\left(8 - 64/12\right) = \boxed{\frac{8\pi}{3}}$$

FOR CYLINDERS

$$2\pi\int_0^2 (2x^2 - x^3)\,dx = 2\pi\left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \boxed{\frac{8\pi}{3}}$$

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20 pts 7. (a) Compute the definite integral

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$

LET $u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}dx = \frac{dx}{2\sqrt{x}}$.

$x=1 \Rightarrow u=1$

$x=4 \Rightarrow u=2$.

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int_1^2 2e^u\,du = 2e^u\Big|_1^2 = \boxed{2(e^2-e)}.$$

20 pts (b) Compute the indefinite integral $\int (\sec 2t)(\tan 2t)\ dt$

SINCE $\frac{d}{dx}\tan(x) = \sec(x)\tan(x)$, LET $u = 2t$

$du = 2dt$

$$\int \sec(2t)\tan(2t)dt$$

$$= \frac{1}{2}\int \sec u \tan u\, du = \frac{1}{2}\sec u + C$$

$$\boxed{= \frac{1}{2}\sec(2t) + C.}$$

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[20 pts] 8. (a) Compute the following indefinite integral $\int p^6 \ln p \, dp$

INTEGRATE BY PARTS. $\left(\int u\,dv = uv - \int v\,du \right)$

$$u = \ln p \qquad dv = p^6 dp$$
$$du = \frac{1}{p} dp \qquad v = \frac{1}{7} p^7$$

$$\int p^6 \ln p \, dp = \frac{1}{7} p^7 \ln p - \frac{1}{7} \int p^6 dp$$
$$= \boxed{\frac{1}{7} p^7 \ln p - \frac{1}{49} p^7 + C}$$

[20 pts] (b) Compute the following definite integral $\int_0^{\pi} t \sin 3t \; dt$.

AGAIN, BY PARTS.

$$u = t \quad dv = \sin 3t \, dt$$
$$du = dt \quad v = \frac{-1}{3} \cos 3t$$

$$\int_0^{\pi} t \sin 3t \, dt = -\frac{1}{3} t \cos(3t) \Big|_0^{\pi} - \int_0^{\pi} \left(-\frac{1}{3} \cos 3t\right) dt$$
$$= -\frac{1}{3} t \cos(3t) + \frac{1}{9} \sin 3t \Big|_0^{\pi}$$
$$= \left(-\frac{1}{3} \pi \cos 3\pi + \frac{1}{9} \sin 3\pi\right) - \left(\frac{-1}{3} 0 \cdot \cos 0 + \frac{1}{9} \sin 0\right)$$
$$= -\frac{1}{3} \pi (-1) + \frac{1}{9} (0) - 0$$
$$= \boxed{\frac{\pi}{3}}$$

Name: SOLUTIONS Id: SPR '10

20 pts 9. (a) Write an equation in Cartesian (x-y) coordinates for the curve with polar equation $r = 2\sin\theta$. Your answer should not contain trigonometric functions.

RECALL THAT $x = r\cos\theta$ $\quad r^2 = x^2+y^2$.
$y = r\sin\theta$ $\quad \tan\theta = y/x$.

MULTIPLY BOTH SIDES BY r TO GET

$$r^2 = 2r\sin\theta$$

SO

$$\boxed{x^2+y^2 = 2y}$$

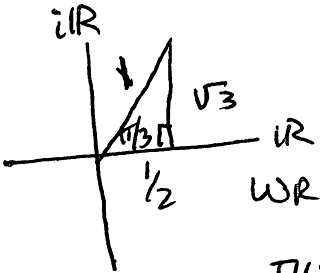
THIS IS OK, BUT WE CAN GET THE FAMILIAR FORM OF A CIRCLE WITH SOME REARRANGING.

$$x^2+y^2-2y = 0$$
$$x^2+y^2-2y+1 = 1$$
$$\boxed{x^2+(y-1)^2 = 1}$$

THIS IS A CIRCLE OF RADIUS 1 AND CENTER $(0,1)$.

20 pts (b) Find the $(a+ib)$-form of the complex number

$$\left[\frac{1+i\sqrt{3}}{2}\right]^{20}$$



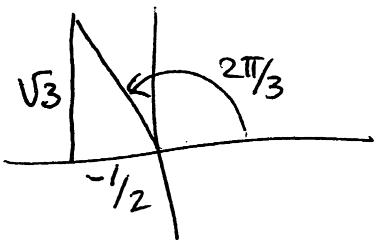
WRITE $\frac{1}{2}+\frac{i\sqrt{3}}{2}$ IN POLAR FORM, AS $e^{i\pi/3}$

THEN

$$\left[\frac{1}{2}+\frac{i\sqrt{3}}{2}\right]^{20} = \left[e^{i\pi/3}\right]^{20} = e^{\frac{20\pi}{3}i} = e^{\left(6\pi+\frac{2\pi}{3}\right)i} = e^{\frac{2\pi i}{3}}$$

CONVERTING BACK TO $a+ib$, WE GET

$$\boxed{e^{\frac{2\pi}{3}i} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i}$$



(6π IS 3 FULL TURNS)

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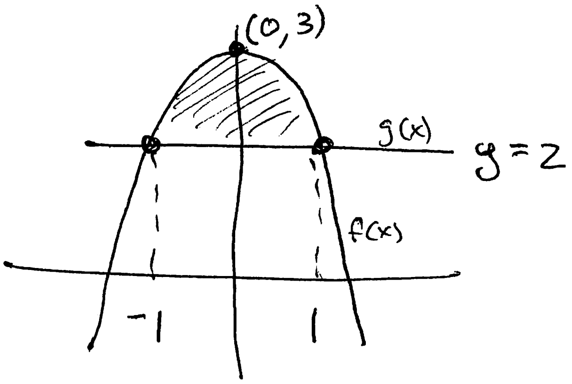
$\boxed{\text{30 pts}}$ 10. Find the center of mass of a flat plate with uniform density that occupies the region bounded by the two curves $y = 2$ and $y = 3 - x^2$.

[THIS TOPIC IS NOT ON THE FALL 2016 EXAM]



NEED TO FIND THE AVERAGE VALUES $\bar{x}$ AND $\bar{y}$

THE TWO CURVES CROSS AT $(-1, 2)$ AND $(1, 2)$

TOTAL AREA IS

$$\int_{-1}^{1} (3-x^2) - 2\,dx = \int_{-1}^{1}(1-x^2)\,dx = x - \frac{x^3}{3}\Big|_{-1}^{1} = \frac{4}{3} = A.$$

$$\bar{x} = \frac{1}{A}\int x\left(f(x)-g(x)\right)dx = \frac{3}{4}\int_{-1}^{1} x - x^3\,dx = \frac{3}{4}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{-1}^{1} = 0$$

$$\bar{y} = \frac{1}{A}\int \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)dx = \frac{3}{8}\int_{-1}^{1}\left((3-x^2)^2 - 2^2\right)dx$$

$$= \frac{3}{8}\int_{-1}^{1}(9 - 6x^2 + x^4 - 4)\,dx$$

$$= \frac{3}{8}\left(5x - 2x^3 + \frac{x^5}{5}\right)\Big|_{-1}^{1}$$

$$= \frac{3}{8}\cdot\frac{32}{5} = \frac{96}{40} = \frac{12}{5}$$

CENTER OF MASS IS $\left(0, \frac{12}{5}\right)$