Name: FALL '18 SOLUTIONS. Id: ____________

1. Evaluate each of the following integrals. If the integral diverges, write "diverges". You must justify your answers for full credit.

10 pts (a) $\int \frac{4x-2}{x^2-x+2}\,dx$ LET $u = x^2 - x + 2$, $du = (2x-1)dx$

$$= \int \frac{2\,du}{u} = 2\ln|u| + C = \boxed{2\ln|x^2-x+2| + C}$$

10 pts (b) $\int_1^\infty \frac{1}{x^3}\,dx$

$$= \lim_{M\to\infty} \left.\frac{x^{-2}}{-2}\right|_1^M = \lim_{M\to\infty} -\frac{1}{2}\left(\frac{1}{M^2} - (1)\right)$$

$$= 0 + \frac{1}{2} = \boxed{1/2}$$

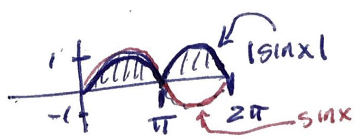
10 pts (c) $\int xe^{4x}\,dx$ BY PARTS: $u = x$ $dv = e^{4x}dx$

$v = dx$ $v = \frac{1}{4}e^{4x}$

$$= \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x}dx$$

$$\boxed{= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C}$$

10 pts (d) $\int_0^{2\pi} |\sin(x)|\,dx$.



$$= \int_0^\pi \sin x\,dx + \int_\pi^{2\pi}(-\sin x)dx$$

$$= -(\cos\pi + \cos(0)) + (\cos(2\pi) - \cos\pi)$$

$$= -(-1) + 1 + 1 - (-1) = \boxed{4}$$

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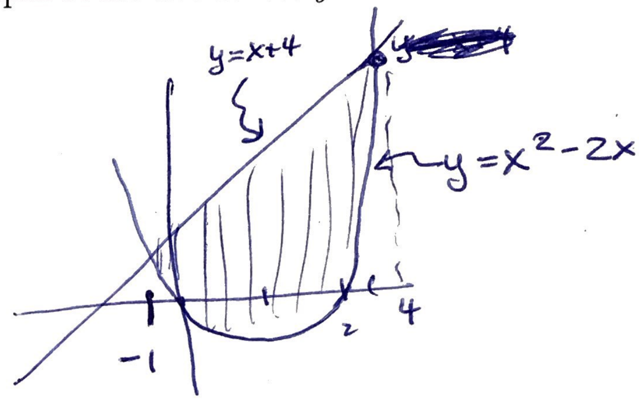
$x(x-2)$

[20 pts] 2. Find the area of the region bounded by the graphs of the two curves $y = x^2 - 2x$ and $y = x + 4$.



WHERE DO CURVES CROSS?

$$x+4 = x^2 - 2x$$

$$\Leftrightarrow \quad 0 = x^2 - 3x - 4 = (x-4)(x+1)$$

SO AT $x = -1$ AND $x = 4$.

AREA IS $\displaystyle\int_{-1}^{4} (x+4) - (x^2 - 2x)\, dx = \int_{-1}^{4} -x^2 + 3x + 4\, dx$

$$= -\frac{x^3}{3} + \frac{3}{2}x^2 + 4x \Big|_{-1}^{4}$$

$$= -\frac{4^3}{3} + \frac{3 \cdot 16}{2} + 16 - \left(\frac{1}{3} + \frac{3}{2} - 4\right)$$

$$= -\frac{64}{3} + \frac{48}{2} + 16 - \frac{1}{3} - \frac{3}{2} + 4$$

ALL ARE OK ANSWERS

I CANT DO ARITHMETIC!

$$\frac{56}{3} + \frac{13}{6} = \boxed{\frac{125}{6}}$$

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10 pts 3. Express the integral $\int_{-2}^{4} x^2 + 2x\,dx$ as a limit of a Riemann sum (with n rectangles). Your final answer should not include symbols like Δx or x_i.

$$\Delta x = \frac{b-a}{n} = \frac{4+2}{n} = \frac{6}{n}$$

$$x_i = -2 + \frac{6i}{n}$$

$$\int_{-2}^{4} (x^2+2x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(-2+\frac{6i}{n}\right)^2 + 2\left(-2+\frac{6i}{n}\right)\right)\cdot \frac{6}{n}}$$

$$\text{OR } \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} 36\frac{i^2}{n^2} - \frac{12i}{n}$$

OR ~~SUCH~~ EQUIVALENT.

15 pts 4. Find the average value of $f(x) = \arctan 3x$ for $0 \le x \le \frac{1}{3}$.

$$\text{AVG VAL} = \frac{1}{3}\int_0^{1/3} \arctan 3x\,dx$$

BY PARTS,
$u = \arctan 3x \quad dv = dx$
$du = \frac{1}{1+9x^2}\cdot 3dx \quad v = x$

$$= \frac{1}{3}\left(x\arctan 3x \Big|_0^{1/3} - \int_0^{1/3} \frac{3x\,dx}{1+9x^2}\right)$$

LET $u = 1+9x^2$, $du = 18x\,dx$
$x=0 \Rightarrow u=1$
$x=1/3 \Rightarrow u=2$

$\arctan(1)=\pi/4$
$\arctan(0)=0$

$$= \frac{1}{3}\left(\frac{1}{3}\cdot\frac{\pi}{4} - \frac{1}{6}\int_1^2 \frac{du}{u}\right)$$

$$= \frac{1}{3}\left(\frac{\pi}{12} - \frac{1}{6}\ln 2 + \frac{1}{6}\ln 1\right) = \boxed{\frac{\pi}{36} - \frac{1}{18}\ln 2}$$

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20 pts 5. Recall that $\int_1^5 \frac{dx}{x} = \ln 5$. Use Simpson's rule with $n = 4$ to write a fraction (or sum of fractions) approximating $\ln 5$.
If you've forgotten Simpson's rule, you can use the Trapezoid rule, but you will lose 5 points for doing so. If you've forgotten that, try using a right-endpoint Riemann sum to maybe get 10 pts.

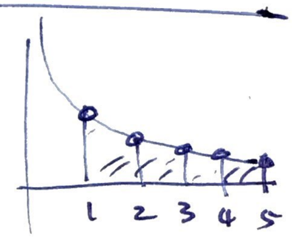
BY SIMPSON'S RULE: $n=4$ GIVES 4 INTERVALS
POINTS ARE $1, 2, 3, 4, 5$

WIDTH

$$\frac{1}{3}\cdot\left(\frac{1}{1} + 4\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 4\cdot\frac{1}{4} + 1\cdot\frac{1}{5}\right)$$

$$= \frac{1 + 2 + \frac{2}{3} + 1 + \frac{1}{5}}{3} = \boxed{\frac{73}{45}} \approx 1.622\ldots$$

BY TRAPEZOID RULE:



$\Delta x = 1$:

$$\frac{1}{2}\left(1+\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}+\frac{1}{5}\right)$$

$$= \frac{1}{2}\left(1 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{2}\left(\frac{101}{30}\right) \approx 1.68333\ldots$$

ANY ARE OK.

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Do **any four** of the five questions 6-10. Cross out the one you don't want graded.

20 pts 6. Compute the following integral. If the integral diverges, write "divergent".

$$\int_2^{4/\sqrt{2}} \frac{dx}{x^2\sqrt{16-x^2}}$$

TO GET RID OF THE $\sqrt{16-x^2}$, WE MAKE THE SUBSTITUTION

$x = 4\sin\theta$ $\qquad x=2 \Rightarrow \frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{3}$

$dx = 4\cos\theta\, d\theta$ $\qquad x = \frac{4}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$

TO GET

$$\int_{\pi/3}^{\pi/4} \frac{4\cos\theta\, d\theta}{(16\sin^2\theta)\sqrt{16-16\sin^2\theta}} = \int_{\pi/3}^{\pi/4} \frac{4\cancel{\cos\theta}\, d\theta}{16\sin^2\theta \cdot 4\cancel{\cos\theta}}$$

$\sqrt{16\cos^2\theta}$

$$= \int_{\pi/3}^{\pi/4} \frac{d\theta}{16\sin^2\theta} = \frac{1}{16}\int_{\pi/3}^{\pi/4} \frac{d\theta}{\sin^2\theta} = \frac{1}{16}\int_{\pi/3}^{\pi/4} \csc^2\theta\, d\theta$$

$\cancel{= \frac{1}{16}\tan\theta \Big|_{\pi/3}^{\pi/4}}$ OOPS

$$= \frac{-1}{16}\cot\theta \Big|_{\pi/3}^{\pi/4}$$

$$= -\frac{1}{16}\left(\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{3}\right)\right)$$

$$= -\frac{1}{16} + \frac{\sqrt{3}}{16} = \boxed{\frac{\sqrt{3}-1}{16}}$$

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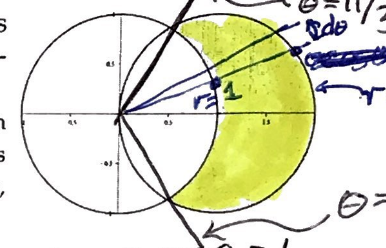
Do **any four** of the five questions 8-12. Cross out the one you don't want graded.

20 pts 11. Find the area of the region that lies inside the circle of radius one centered at $(1, 0)$, but outside the circle of radius one centered at the origin.

You can do this either in polar coordinates (where the two curves are given by $r = 2\cos\theta$ and $r = 1$) or in rectangular coordinates (where the curves are given by $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$). In the rectangular case, you need to cut up the area appropriately.



IN POLAR COORDS: CURVES CROSS WHEN $2\cos\theta = 1$

IE $\cos\theta = 1/2$

IE $\theta = \pm\pi/3$

AREA IS $\frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 - 1^2 \, d\theta$

$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1 \, d\theta$

$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2(1+\cos 2\theta) - 1 \, d\theta$

$= \frac{1}{2}(\theta + \sin 2\theta)\Big|_{-\pi/3}^{\pi/3} = \frac{\pi}{3} + \sin(2\pi/3)$

$= \boxed{\pi/3 - \frac{\sqrt{3}}{2}}$

IN RECT. COORDS,

SPLIT IT INTO PART WITH $x<1$ (BLACK) AND $x>1$ (WHITE) THEN DOUBLE IT.

FOR $x<1$,

UPPER CURVE IS $y = \sqrt{1-(x-1)^2}$, LOWER IS $y = \sqrt{1-x^2}$.

THEY CROSS AT $x = 1/2$.

GET $\underbrace{\int_{1/2}^{1} \sqrt{1-(x-1)^2} - \sqrt{1-x^2}\, dx}_{= \sqrt{3}/4 - \pi/12}$

OTHER PIECE IS A $\frac{1}{4}$ CIRCLE, $A = \pi/4$.

SO ADD THEM TOGETHER, DOUBLE IT, YOU GET SAME ANSWER

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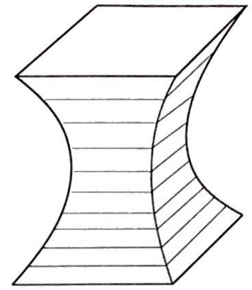
Do **any four** of the five questions 6-10. Cross out the one you don't want graded.

| 20 pts |

8. A pillar that is π feet tall is made so that every horizontal cross section at height h is a square of side length
$$s(h) = 3 - \sin(h).$$



(a) Write an integral which represents the volume of the pillar.

$$V = \int_0^{\pi} \text{AREA(SLICE)}\, dh$$

$$= \int_0^{\pi} (3 - \sin h)^2\, dh$$

(b) Evaluate the integral to find the volume of the pillar.

$$\int_0^{\pi} 9 - 6\sin(h) + \sin^2(h)\, dx$$

$$= 9\pi + (6\cos(h))\Big|_0^{\pi} + \int_0^{\pi} \sin^2(h)\, dh$$

$$= 9\pi + 6(-1-1) + \int_0^{\pi} \tfrac{1}{2}(1 + \cos 2h)\, dx$$

$$= 9\pi - 12 + \tfrac{1}{2}\left(\frac{h}{2} + \frac{\sin 2h}{2}\Big|_0^{\pi}\right)$$

$$= 9\pi - 12 + \frac{\pi}{4} + 0$$

$$= \frac{35\pi}{4} - 12$$

EITHER IS FINE.

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Do **any four** of the five questions 6-10. Cross out the one you don't want graded.

20 pts 9. Compute the following integral. If the integral diverges, write "divergent".

$$\int_1^2 \frac{x^2+2x+3}{x^2(3-x)}\,dx$$

WE WILL NEED TO DO PARTIAL FRACTIONS. NOTE THAT THE X^2 TERM IN THE DENOMINATOR MEANS WE WILL NEED THREE CONSTANTS:

$$\frac{X^2+2x+3}{x^2(3-x)} = \frac{Ax+B}{x^2} + \frac{C}{3-x} \iff X^2+2x+3 = (Ax+B)(3-x) + Cx^2$$

LET $X=0$: $0+3 = (0+B)(3)+0 = 3B \Rightarrow \boxed{B=1}$

LET $X=3$: $9+6+3 = 0+9\cdot C \Rightarrow 18 = 9C \Rightarrow \boxed{C=2}$

NOW WE HAVE $x^2+2x+3 = (Ax+1)(3-x)+2x^2$

PICK ANY OTHER X!

LET $X=1$: $1+2+3 = (A+1)(2)+2$

$\Rightarrow 2 = A+1 \Rightarrow \boxed{A=1}$

SO WE HAVE

$$\int_1^2 \frac{x^2+2x+3}{x^2(3-x)}\,dx = \int_1^2 \left(\frac{x+1}{x^2}+\frac{2}{3-x}\right)dx = \int_1^2 \frac{1}{x}+\frac{1}{x^2}+\frac{2}{3-x}\,dx$$

$$= \ln|x| - \frac{1}{x} - 2\ln|3-x| \Big|_1^2$$

$$= \left(\ln 2 - \frac{1}{2} + \underbrace{2\ln(1)}_{=0}\right) - \left(\underbrace{\ln(1)}_{=0} - 1 - 2\ln 2\right)$$

$$= \boxed{3\ln 2 + \frac{1}{2}}$$

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Do **any four** of the five questions 6-10. Cross out the one you don't want graded.

20 pts 10. After a certain treatment, the probability of the symptoms of a specific illness recurring t years after the treatment is given by a log-logistic distribution[1] of the form

$$f(t) = \begin{cases} \dfrac{18t}{(t^2+9)^2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

After the treatment, what is the probability that the illness will not recur for at least 5 years?

THE PROBABILITY THE DISEASE DOES NOT RECUR IN 5 YRS

IS $P(t>5) = \int_5^\infty \frac{18t}{(t^2+9)^2}\,dt$ (OR $1-P(t<5) = 1-\int_0^5 \frac{18t}{(t^2+9)^2}$ IF YOU PREFER

LET $u=t^2+9$, $du = 2t\,dt$

$t=5 \Rightarrow u=34$, $t=\infty \Rightarrow u=\infty$

$$= \int_{34}^\infty \frac{9}{u^2}\,du = -\frac{9}{u}\Big|_{34}^\infty$$

$$= \left(\lim_{u\to\infty} \frac{-9}{u}\right) - \left(\frac{-9}{34}\right)$$

$$= 0 + \frac{9}{34} = \boxed{\frac{9}{34}}$$

(A BIT MORE THAN 25%)

[1]This is also known as a Fisk distribution when used in economics. This distribution depends on two parameters: a scale parameter α and a shape parameter β; for this treatment, $\alpha = 3$ and $\beta = 2$.