

Name: SEWTONS

Id: FALL '16

10 pts 1. Calculate the integral $\int_{-1}^1 x^3 + 3x^2 - \frac{1}{1+x^2} dx$.

If the integral is improper and it does not converge, write "Diverges" (and justify).

$$= \left. \frac{1}{4}x^4 + x^3 - \arctan(x) \right|_{-1}^1$$

$$= \left(\frac{1}{4} + 1 - \frac{\pi}{4} \right) - \left(\frac{1}{4} - 1 + \frac{\pi}{4} \right)$$

$$= \boxed{2 - \frac{\pi}{2}}$$

10 pts 2. Calculate the indefinite integral $\int we^{2w} dw$.

BY PARTS: $\left. \begin{array}{l} u = w \quad dv = e^{2w} dw \\ du = dw \quad v = \frac{1}{2}e^{2w} \end{array} \right\} = \frac{1}{2}we^{2w} - \frac{1}{2}\int e^{2w} dw$

$$= \boxed{\frac{1}{2}we^{2w} - \frac{1}{4}e^{2w} + C}$$

10 pts 3. Calculate the integral $\int_1^\infty \frac{du}{(2u-1)^2}$.

If the integral is improper and it does not converge, write "Diverges" (and justify).

LET $w = 2u - 1$ WHEN $u = 1, w = 1$
 $dw = 2du$ WHEN $u = \infty, w = \infty$

$$= \frac{1}{2} \int_1^\infty \frac{dw}{w^2} = \left. -\frac{1}{2w} \right|_1^\infty = \lim_{w \rightarrow \infty} -\frac{1}{2w} + \frac{1}{2}$$

$$= 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

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15 pts 4. Calculate the indefinite integral $\int \frac{3dx}{(2x+1)(x-1)} = \int \left(\frac{A}{2x+1} + \frac{B}{x-1} \right) dx$

$$3 = A(x-1) + B(2x+1)$$

IF $x=1$, $3 = 0 + 3B$, so $B=1$.

IF $x=-\frac{1}{2}$, $3 = A \cdot \frac{3}{2} + 0$, so $A=2$

ALTERNATIVELY, WE HAVE

$$3 = Ax - A + 2Bx + B, \text{ so } A + 2B = 0$$

$$-A + B = 3$$

ADDING GIVES $3B = 3$, so $A = -2$
 $B = 1$

$$= \int \frac{-2}{2x+1} + \frac{3}{x-1} dx = -\ln|2x+1| + \ln|x-1| + C$$

15 pts 5. Calculate the indefinite integral $\int \sin^4(2t) \cos^3(2t) dt$

REWRITE USING $\cos^2(2t) = 1 - \sin^2(2t) + 1$

so $\int \sin^4(2t) (1 - \sin^2(2t)) \cos(2t) dt$

$$= \int \sin^4(2t) \cos(2t) - \sin^6(2t) \cos(2t) dt$$

LET $u = \sin 2t$, so $du = 2 \cos 2t$

$$= \frac{1}{2} \int u^4 - u^6 du = \frac{1}{2} \left(\frac{u^5}{5} - \frac{u^7}{7} \right) + C$$

$$= \frac{1}{2} \left(\frac{\sin^5(2t)}{5} - \frac{\sin^7(2t)}{7} \right) + C$$

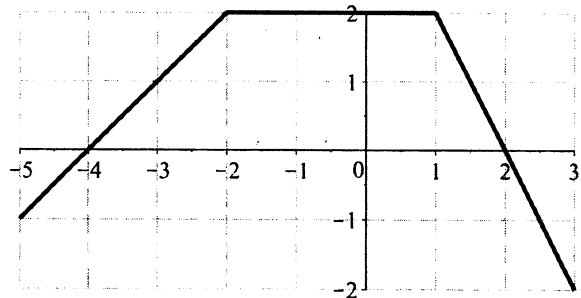
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6. The function $f(x)$ has the graph shown at right, for $-5 \leq x \leq 3$.

Define another function $g(x)$ by

$$g(x) = \int_{-5}^x f(t) dt.$$



5 pts

- (a) For what x does $g(x)$ take on its maximum value for $-5 \leq x \leq 3$?
(If there is no maximum, write "None"; if there are several such x , list them all.)

SINCE CRITICAL POINTS ARE WHERE $f(t)=0$,
CHOICES ARE $-5, -4, 2$, AND 3 .

2 IS A LOCAL MAXIMUM, AND ALSO
THE GLOBAL ONE.

5 pts

- (b) What is $g(2)$? (If it is not defined, write "DNE".)

COUNTING BOXES, WE GET

$$g(2) = \boxed{9.5} = \frac{19}{2}.$$

5 pts

- (c) What is the minimum value of $g(x)$ for $-5 \leq x \leq 3$?
(If there is no minimum, write "None".)

$g(-5) = -\frac{1}{2}$ IS THE SMALLEST $g(x)$
TAKES ON IN THE RANGE.

5 pts

- (d) What is the largest interval (for x between -5 and 3) on which $g(x)$ is concave down?
(If there is no such interval, write "None".)

NOTE THAT $g''(x) = f'(x)$,

SO $g''(x)$ IS POSITIVE FOR $-5 < x < -2$
ZERO FOR $-2 < x < 1$

AND NEGATIVE FOR $1 < x < 3$.

ANSWER $\boxed{1 < x < 3}$

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20 pts

7. Consider the region lying above the graph of $y = x$, below the graph of $y = 2x \cos(x)$, to the right of $x = 0$, and to the left of $x = \frac{\pi}{2}$. See the figure at right (note that the region does not contain all x values between 0 and $\frac{\pi}{2}$).

Find the area of this region.

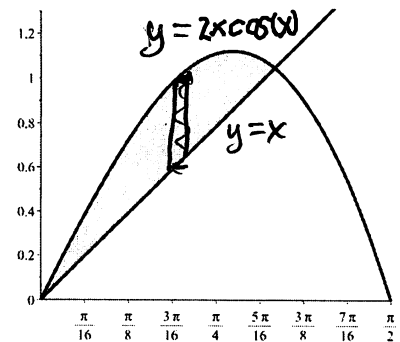
WHERE DO THEY CROSS?

$$2x \cos x = x$$

$$\text{IF } x=0 \text{ OR } \cos x = \frac{1}{2}$$

$$\Rightarrow x = \frac{\pi}{3}$$

$$\text{AREA IS } \int_0^{\pi/3} (2x \cos x - x) dx$$



$$u = x \quad v = \sin(x)$$

$$du = dx \quad dv = \cos(x)$$

$$-\left(\frac{y^2}{2}\right) \Big|_0^{\pi/3}$$

$$\left(x \sin x - \int \sin(v) dx\right) - \left(\frac{\pi}{3}\right)^2 - 0$$

$$\left(x \sin x + \cos x\right) \Big|_0^{\pi/3} - \frac{\pi^2}{18}$$

$$\left(\frac{\pi}{3} \cdot \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right) - (0 + \cos 0) - \frac{\pi^2}{18}$$

$$\frac{\pi}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} - 1 - \frac{\pi^2}{18}$$

$$\frac{\pi\sqrt{3} + 3}{6} - 1 - \frac{\pi^2}{18}$$

$$\frac{\pi\sqrt{3} - 3}{6} - \frac{\pi^2}{18}$$

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Do **any four** of the five questions 8-12. Cross out the one you don't want graded.

20 pts

8. A bottle 12 inches high has its radius r (in inches) at height h recorded in the table at right. Use **Simpson's rule** to estimate the volume of the bottle.

(If you don't remember Simpson's rule, you may use the Trapezoid rule instead, but you will get a maximum of 15 points. Or, use left or right endpoints for half credit. **Clearly indicate which method you are using!**)

h	0	3	6	9	12
r	2	3	3	2	1

Hint: First write an integral representing the volume in terms of some function $r(h)$, then use Simpson's rule to approximate it.

SLICE HORIZONTALLY.

A TYPICAL CROSS SECTION AT h
IS A DISK OF RADIUS $r(h)$.

AREA IS $\pi(r(h))^2$

SO VOLUME IS

$$\int_0^{12} \pi (r(h))^2 dh$$

WE KNOW $r(h)$ AT 0, 3, 6, 9, 12

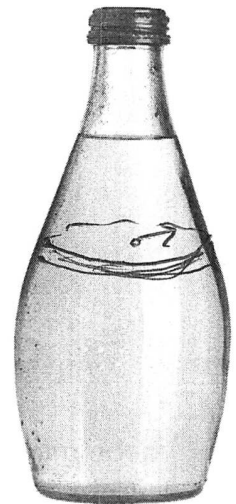
SO WE TAKE $n=4$ FOR SIMPSONS. $b-a=12$.

THEN WE GET

$$\frac{12}{4 \cdot 3} (\pi) (1 \cdot 2^2 + 4 \cdot 3^2 + 2 \cdot 3^2 + 4 \cdot 2^2 + 1 \cdot 1^2)$$

$$= \pi (4 + 36 + 18 + 16 + 1)$$

$$= \boxed{75 \pi \text{ in}^3}$$



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10 pts

9. (a) Write an integral that represents the arc length of the graph of $y = 2x + \frac{x^3}{3}$ for $0 \leq x \leq 3$. You do not need to calculate the value of the integral!

RECALL THAT THE ARC LENGTH OF $y = f(x)$ FROM a TO b IS $\int_a^b \sqrt{1 + (f'(x))^2} dx$

SO:

$$AL = \int_0^3 \sqrt{1 + (2 + x^2)^2} dx$$

10 pts

- (b) Find the average value of the function $f(x) = x^2 \sin(2x)$ over the domain $0 \leq x \leq \frac{\pi}{2}$.

AVERAGE VALUE IS $\frac{1}{b-a} \int_a^b x^2 \sin(2x) dx$

$$\frac{2}{\pi} \int_0^{\pi/2} x^2 \sin(2x) dx$$

$$\begin{array}{l} u = x^2 \quad dv = \sin(2x) dx \\ du = 2x dx \quad v = -\frac{1}{2} \cos(2x) \end{array}$$

$$= \frac{2}{\pi} \left(-\frac{1}{2} x^2 \cos 2x \Big|_0^{\pi/2} + \int_0^{\pi/2} x \cos 2x dx \right)$$

$$\begin{array}{l} u = x \quad dv = \cos 2x dx \\ du = dx \quad v = \frac{1}{2} \sin 2x \end{array}$$

$$= \frac{2}{\pi} \left(-\frac{1}{2} x^2 \cos(2x) \Big|_0^{\pi/2} + \frac{1}{2} x \sin(2x) \Big|_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \sin 2x dx \right)$$

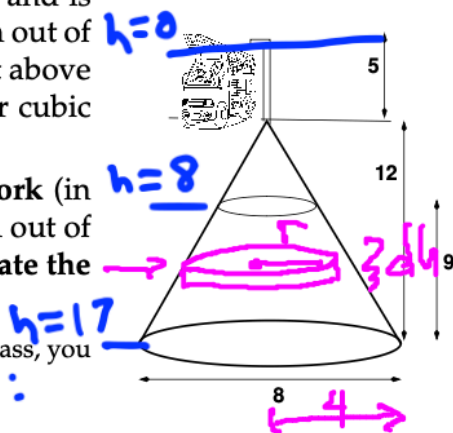
$$= \frac{2}{\pi} \left(-\frac{1}{2} \cdot \frac{\pi^2}{4} (-1) + 0 - \frac{1}{4} \cos(2x) \Big|_0^{\pi/2} \right) = \frac{\pi}{4} - \frac{1}{\pi}$$

20 pts 10. Fireman Fred has an underground tank partially full of Fluorotelomer Fire-Fighting Foam.

The tank is conical, with the vertex at the top of the tank. The height of the tank is 12 feet, with a diameter of 8 feet, and is filled to a height of 9 feet. Fred wants to pump the foam out of the tank and into his truck, which fills at a height 5 feet above ground level. The foam has a density of 4 pounds per cubic foot and has a delightful minty scent.

Write an integral which represents the amount of work (in foot-pounds) required for Fred to pump all of the foam out of the tank and into his truck. (You do not need to calculate the integral).

Since the weight of the foam is given (in pounds) instead of the mass, you should not include g , the acceleration due to gravity.



FIRST, LETS LET h REPRESENT THE DISTANCE FROM THE TOP OF THE SPOUT TO THE FOAM BEING REMOVED, SO WE CALCULATE THE WORK REMOVING FOAM FROM $h=8$ TO $h=17$.

WE NEED THE FORCE (OR THE WEIGHT) OF A DISK OF THICKNESS dh AT SOME HEIGHT h . THIS IS A ~~SMALL~~ ^{TINY} CYLINDER OF THICKNESS dh AND RADIUS $r(h)$.

NOTE THAT SINCE THE TANK IS CONICAL, THE RADIUS GROWS LINEARLY. AT $h=5$, $r=0$ AND AT $h=17$, $r=4$. SO $r(h) = (h-5)/3$, AND THE WEIGHT WILL BE $\pi r^2 dh = \frac{\pi (h-5)^2}{9} dh$.

THIS WEIGHT (= FORCE) MUST BE APPLIED OVER A DISTANCE OF h . SO

$$\text{WORK} = \int_8^{17} \frac{h\pi}{9} (h-5)^2 dh$$

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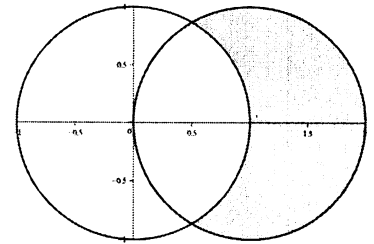
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20 pts

11. Find the area of the region that lies inside the circle of radius one centered at (1, 0), but outside the circle of radius one centered at the origin.

You can do this either in polar coordinates (where the two curves are given by $r = 2 \cos \theta$ and $r = 1$) or in rectangular coordinates (where the curves are given by $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$). In the rectangular case, you need to cut up the area appropriately.



IN POLAR COORDS: CURVES CROSS WHEN $2 \cos \theta = 1$
 i.e. $\cos \theta = 1/2$
 i.e. $\theta = \pm \pi/3$

$$\begin{aligned}
 \text{AREA IS } & \frac{1}{2} \int_{-\pi/3}^{\pi/3} (2 \cos \theta)^2 - 1^2 d\theta \\
 &= \frac{1}{2} \int_{-\pi/3}^{\pi/3} 4 \cos^2 \theta - 1 d\theta \\
 &= \frac{1}{2} \int_{-\pi/3}^{\pi/3} 2(1 + \cos 2\theta) - 1 d\theta \\
 &= \frac{1}{2} (\theta + \sin 2\theta) \Big|_{-\pi/3}^{\pi/3} = \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) \\
 &= \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{2}}
 \end{aligned}$$

IN RECT. COORDS,



SPLIT IT INTO PART WITH $x < 1$ (BLACK) AND $x > 1$ (WHITE) THEN DOUBLE IT.

FOR $x < 1$,

UPPER CURVE IS $y = \sqrt{1 - (x-1)^2}$, LOWER IS $y = \sqrt{1 - x^2}$.
 THEY CROSS AT $x = 1/2$.

GET $\int_{1/2}^1 \sqrt{1 - (x-1)^2} - \sqrt{1 - x^2} dx$
 $= \sqrt{3}/4 - \pi/12$

OTHER PIECE IS A $\frac{1}{4}$ CIRCLE,

$A = \pi/4$.

SO ADD THEM TOGETHER, DOUBLE IT, YOU GET SAME ANSWER

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Do **any four** of the five questions 8-12. Cross out the one you don't want graded.

12. Wind speed can be modeled by the Rayleigh distribution $f(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ where σ is the most common wind speed (the mode).

10 pts

- (a) If the most common wind speed is a breeze of 5 knots, what is the probability that the wind speed will be more than 20 knots? Do not approximate e , logs, or square roots.

$\sigma = 5$, AND $P(\text{SPEED} > 20)$ IS GIVEN BY

$$\int_{20}^{\infty} \frac{x}{25} e^{-x^2/50} dx$$

LET $u = +x^2/50$
 SO $du = +x/25 dx$

$x=20 \Rightarrow u = \frac{400}{50} = 8$
 $x=\infty \Rightarrow u = \infty$

$$= \int_8^{\infty} e^{-u} du$$

$$= -e^{-u} \Big|_8^{\infty} = \lim_{b \rightarrow \infty} (-e^{-b} + e^{-8}) = 0 + e^{-8} = \boxed{\frac{1}{e^8}}$$

10 pts

- (b) Again assuming $\sigma = 5$, calculate the median¹ wind speed. Do not give an approximation.

MUST SOLVE

$$\int_0^m \frac{x}{25} e^{-x^2/50} dx = \frac{1}{2}$$

$$= \int_0^{m^2/50} e^{-u} du$$

$$= -e^{-u} \Big|_0^{m^2/50} = -e^{-m^2/50} + 1$$

SO $e^{-m^2/50} = 1/2$
 TAKE LN OF BOTH SIDES

$$-m^2/50 = \ln 2$$

$$m^2 = 50 \ln 2$$

$$m = \sqrt{50 \ln 2}$$

¹Recall that the **median** of a probability density function $f(x)$ is the number m so that the probability that $x \geq m$ is $\frac{1}{2}$ (and so also the probability that $x < m$ is $\frac{1}{2}$).