MAT 126 Solutions to Midterm 2 (Space)

1. Compute each of the following definite integrals. If the integral does not converge, write "Diverges". In all cases you must justify your answer to receive full credit.

10 pts (a) $\displaystyle\int_2^\infty \frac{dx}{x^2}$

Solution:

$$\int_2^\infty \frac{dx}{x^2} = \lim_{M\to\infty}\int_2^M x^{-2}\,dx = \lim_{M\to\infty}\left.\frac{x^{-1}}{-1}\right|_2^M = \lim_{M\to\infty}\left(\frac{-1}{M}+\frac{1}{2^1}\right) = 0+\frac{1}{2} = \boxed{\frac{1}{2}}.$$

10 pts (b) $\displaystyle\int_0^{\pi/2} \sin(y)\cos^3(y)\,dy$

Solution: Make the substitution $u=\cos(y)$ so that $du=-\sin(y)\,dy$. Also, when $y=0$ we have $u=\cos(0)=1$, and when $y=\pi/2$, $u=\cos(\pi/2)=0$. Then the integral becomes

$$-\int_1^0 u^3\,du = \left.\frac{u^4}{4}\right|_0^1 = \frac{1}{4}-0 = \boxed{\frac{1}{4}}.$$

10 pts (c) $\displaystyle\int_0^4 \frac{dz}{(z-1)^5}$

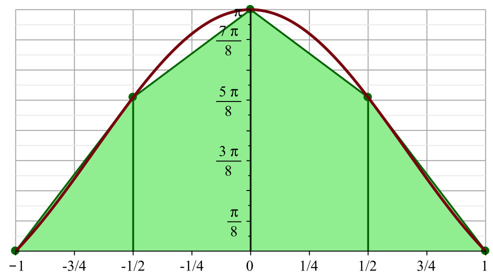
Solution: $\frac{1}{(z-1)^5}$ is not defined when $z=1$, so we must split the integral at $z=1$:

$$\begin{aligned}\int_0^1 \frac{dz}{(z-1)^5}+\int_1^4 \frac{dz}{(z-1)^5} &= \lim_{b\to1^-}\int_0^b \frac{dz}{(z-1)^5}+\lim_{a\to1^+}\int_a^4 \frac{dz}{(z-1)^5}\\ &= \lim_{b\to1^-}\left(\frac{1}{4(b-1)^4}-\frac{1}{4(-1)^4}\right)+\lim_{a\to1^+}\left(\frac{1}{4(3)^4}-\frac{1}{4(a-1)^4}\right).\end{aligned}$$

Since $\lim\limits_{a\to1^+}(1/(a-1)^4$ and $\lim\limits_{b\to1^-}(1/(b-1)^4$ both diverge, $\boxed{\text{the integral }\textbf{diverges}}$.

$\boxed{\text{30 pts}}$ 2. At right is shown the graph of the function

$$f(x) = \begin{cases} \frac{\sin \pi x}{x} & x \neq 0 \\ \pi & x = 0 \end{cases} .$$



(a) Use the trapezoid method with 4 intervals to approximate the value of $\displaystyle\int_{-1}^{1} f(x)\,dx$.

Solution: In case you didn't memorize the formula (I never do), remember from middle school that the area of a trapezoid is the length of the base times the average of the heights of the two sides. Each base is of length $1/2$, so we have

$$\frac{1}{2}\left(\frac{1}{2}(0+2) + \frac{1}{2}(2+\pi) + \frac{1}{2}(\pi+2) + \frac{1}{2}(2+0)\right) = \frac{1}{4}(4\cdot 2 + 2\pi) = \boxed{2 + \frac{\pi}{2}} .$$

You might have used $f(\pm 1/2) = 5\pi/8$, but this is only close: $f(1/2) = 2$, and $5\pi/8 \approx 1.9635$.

(b) Estimate the error[1] in your answer in the previous part. You might find it helpful to know that $|f'(x)| < 13/3$ or that $-21/2 < -\pi^3/3 \leq f''(x) \leq 2\pi < 13/2$ for $|x| \leq 1$.

Solution: Note that we want $K = \max|f''(x)|$, and since $|-\pi^3/3| > |2\pi|$, we must use the absolute value of negative bound on f''. Taking $K = 21/2$ gives us

$$E_T < \frac{21/2 \cdot 2^3}{12 \cdot 4^2} = \boxed{\frac{7}{16}} = 0.4375 \ .$$

We get a slightly better estimate using $K = \pi^3/3$ instead: $\boxed{\frac{\pi^3}{72}} \approx .4307$. Either is fine.

The actual error is the approximation from the first part is closer to 0.133078. But you can't know that without computing the integral to fairly high precision first.

(c) How many intervals are required to ensure that the error is less than $1/100$ when using the trapezoid rule to estimate this integral?

Solution: Taking $K = 21/2$ as above, we need to find n so that $\dfrac{21/2 \cdot 8}{12n^2} < \dfrac{1}{100}$. Rewriting, we want

$$n^2 > 100 \cdot \frac{4 \cdot 21}{12} = 700 \quad \text{that is} \quad n > 10\sqrt{7} \approx 26.46 \ .$$

So we should take $\boxed{n \geq 27}$.

[1]The max. error in the trapezoid approximation to $\int_a^b f(x)dx$ using n intervals is $\frac{K(b-a)^3}{12n^2}$, where $K \geq |f''(x)|$ for all $a \leq x \leq b$. Also, $\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$, $\sqrt{7} \approx 2.646$, $\sqrt{13} \approx 3.606$ and $\sqrt{21} \approx 4.583$.

15 pts

3. Evaluate the indefinite integral $\displaystyle\int \frac{2\,dx}{(x+1)(x^2+1)}$.

   Use K for an unknown constant. You must show all steps to receive credit.

Solution: We need to use partial fractions. Write $\dfrac{2\,dx}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$, then cross-multiply to obtain $A(x^2+1)+(Bx+C)(x+1) = 2$. We can then find A, B and C either by equating coeeficients (I don't like to do that) or by choosing values of x and solving.

- Let $x = -1$ to obtain $A(2) + 0 = 2$, so $A = 1$.
- Now we have $x^2+1+(Bx+C)(x+1) = 2$. Taking $x = 0$ gives $(0+C)(1) = 1$ so $C = 1$.
- This leaves us with $x^2 + (Bx+1)(x+1) = 1$. Take $x = 1$ to get $1 + (B+1)(2) = 1$, so $B + 1 = 0$ and hence $B = -1$.

This means we have

$$\int \frac{2\,dx}{(x+1)(x^2+1)} = \int \frac{1}{x+1} + \frac{-x+1}{x^2+1}\,dx = \int \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$$
$$= \boxed{\ln|x+1| - \tfrac{1}{2}\ln(x^2+1) + \arctan(x) + K} \; .$$

15 pts

4. Find the average value of $f(x) = x^2$ for $-2 \le x \le 1$.

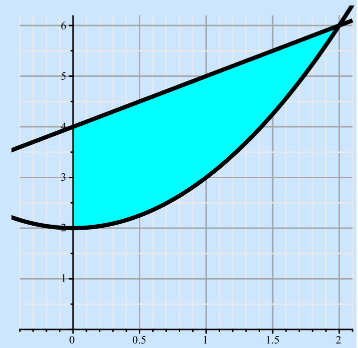
Solution:

$$\frac{1}{1-(-2)} \int_{-2}^{1} x^2\,dx = \frac{1}{3}\,\frac{x^3}{3}\bigg|_{-2}^{1} = \frac{1}{3\cdot 3}\,(1+8) = \frac{9}{9} = \boxed{1} \; .$$

5. Let R be the region with $x \geq 0$ and bounded by the graphs of $y = x^2 + 2$ and $y = x + 4$.

15 pts (a) Calculate the area of the region R.

Solution: The graph of the region R is shown at right.
The graphs cross when $x^2 + 2 = x + 4$, that is when $x = -1$ and $x = 2$. Since we want $x \geq 0$, the area of R is given by



$$\int_0^2 (x+4) - (x^2+2)\, dx = \int_0^2 -x^2 + x + 2\, dx$$
$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_0^2$$
$$= -\frac{8}{3} + \frac{4}{2} + 4 = 6 - \frac{8}{3} = \boxed{\frac{10}{3}}\ .$$

15 pts (b) Calculate the volume of the solid obtained by rotating R around the x-axis.

Solution: An image of the solid is shown below.
It is easiest to slice this by varying x; a typical slice is a washer with inner radius $x^2 + 2$ and outer radius $x + 4$. The volume of a washer of thickness dx at a given x value will be

$$\pi\left((x+4)^2 - (x^2+2)^2\right)\ dx\ \ ,$$

so the volume is given by the integral

$$\int_0^2 \pi\left((x+4)^2 - (x^2+2)^2\right)\ dx = \pi \int_0^2 12 + 8x - 3x^2 - x^4\, dx = \frac{128\pi}{5}$$