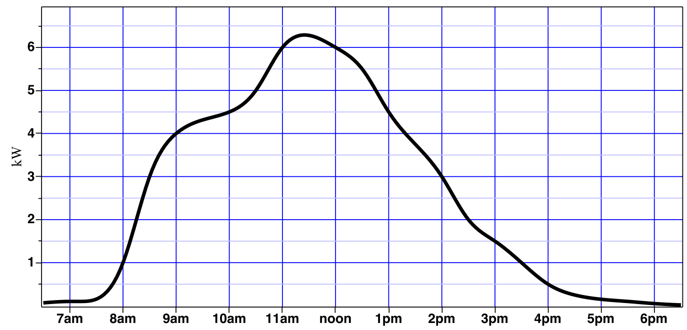
Math 126 Solutions to Midterm 1 (Jean)

20 pts

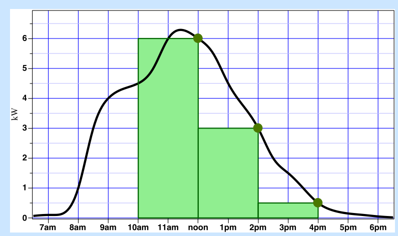
1. At right is shown a graph of the amount of electricity generated (in kilowatts) by a home solar array on a recent day, as a function of time. Let $E(t)$ denote this function. Then the total power produced (in kilowatt-hours) between times $t = a$ and $t = b$ is given by the integral $\int_a^b E(t)\,dt$.



   (a) Use a Riemann sum with three intervals of equal width evaluated at **the right endpoint**, to calculate the total power generated between 10am and 4pm.

Solution:
Using three rectangles evaluated on the right, since the timespan is 6 hours, the width of each is 2h. This gives

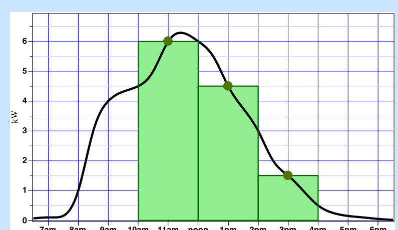
$$2 \cdot (6 + 3 + 0.5) = 19 \text{ kilowatt-hours.}$$



   (b) Use a Riemann sum with three intervals of equal width evaluated at **the midpoint**, to calculate the total power generated between 10am and 4pm.

Solution:
Using three rectangles evaluated at their midpoints where the width of each is 2h, we obtain

$$2 \cdot (6 + 4.5 + 1.5) = 24 \text{ kilowatt-hours.}$$



2. Evaluate each of the indefinite integrals below.

5 pts

   (a) $\int e^{1-3t}\,dt$

Solution: Make the substitution $u = 1 - 3t$, so $du = -dt$. Thus we have

$$\int e^{1-3t}\,dt = -\int e^u\,du = -e^u + C = \boxed{C - e^{1-3t}}\;.$$

5 pts (b) $\displaystyle\int \theta \sin 3\theta \, d\theta$

Solution: Integrating by parts with $u = \theta$ and $dv = \sin 3\theta \; d\theta$, we obtain $du = d\theta$ and $v = \int dv = -\cos 3\theta/3$. This yields

$$\int \theta \sin 3\theta \, d\theta = -\frac{\theta \cos 3\theta}{3} + \frac{1}{3}\int \cos 3\theta \, d\theta = \boxed{C - \frac{\theta \cos 3\theta}{3} + \frac{\sin 3\theta}{9}}.$$

3. Evaluate each of the definite integrals below.

5 pts (a) $\displaystyle\int_0^{\sqrt{3}} \frac{7x+1}{1+x^2} \, dx$

Solution:

$$\begin{aligned} \int_0^{\sqrt{3}} \frac{7x+1}{1+x^2} \, dx &= \int_0^{\sqrt{3}} \frac{7x \, dx}{1+x^2} + \int_0^{\sqrt{3}} \frac{dx}{1+x^2} \\ &= 7\int_1^4 \frac{du/2}{u} + \arctan(x)\Big|_0^{\sqrt{3}} = \tfrac{7}{2}\ln|u|\Big|_1^4 + \left(\frac{\pi}{3} - 0\right) = \boxed{\tfrac{7}{2}\ln|4| + \frac{\pi}{3}}, \end{aligned}$$

with the substitution $u = 1 + x^2$, $du = 2x \, dx$ in the first integral after pulling out the 7.

5 pts (b) $\displaystyle\int_{-1}^{4} \left|w^3\right| \, dw$

Solution: Recall that $|w^3| = \begin{cases} -w^3 & \text{if } w < 0 \\ w^3 & \text{for } w \geq 0 \end{cases}$. Split the integral at zero to obtain

$$\begin{aligned} \int_{-1}^{4} \left|w^3\right| \, dw &= -\int_{-1}^{0} w^3 \, dw + \int_0^4 w^3 \, dw = -\frac{w^4}{4}\Big|_{-1}^{0} + \frac{w^4}{4}\Big|_0^4 \\ &= -\left(0 - \frac{1}{4}\right) + \left(\frac{4^4}{4} - 0\right) = \boxed{\frac{1+4^4}{4} = \frac{257}{4}}. \end{aligned}$$

5 pts (c) $\displaystyle\int_1^4 \ln(1+z) \, dz$

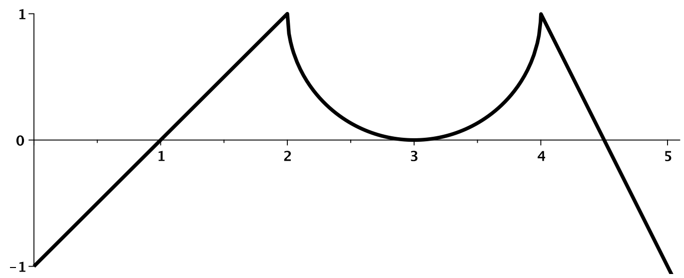
Solution: It isn't strictly necessary, but let's make the substitution $x = 1 + z$ (so $dx = dz$) and the integral becomes $\displaystyle\int_2^5 \ln(x) \, dx$. Then integrate by parts with $u = \ln(x)$ and $dv = dx$. This gives us $du = dx/x$ and $v = x$ (so $v \, du = dx$).

$$\int_2^5 \ln(x) \, dx = x\ln(x)\Big|_2^5 - \int_2^5 dx = (5\ln 5 - 2\ln 2) - (5 - 2) = \boxed{5\ln 5 - 2\ln 2 - 3}.$$

4. Let $g(t)$ be the function given by

$$g(t) = \begin{cases} t-1 & t \le 2 \\ 1-\sqrt{1-(t-3)^2} & 2 < t \le 4 \\ 9-2t & t > 4 \end{cases}$$

with its graph shown at right. Let



$$F(x) = \int_0^x g(t)\,dt\,.$$

5 pts

(a) For what values of x between 0 and 5 is $F(x) \le 0$?

Solution: Observe that the (signed) area between the graph and the axis for $0 < x < 1$ is a triangle with base 1, height -1, so this area is $-\frac{1}{2}$. Similarly, the area for x between 1 and 2 is $-\frac{1}{2}$, so $F(2) = 0$, and for all $x < 2$, $F(x)$ is negative.
Notice that $g(t) > 0$ for $2 < x < 4.5$, and by an argument similar to the above, the total area for $x \in [3, 5]$ is 0. This means $F(x)$ is positive for $x \in [2, 5]$, and so

$$\boxed{F(x) \le 0 \text{ for } 0 \le x \le 2} \text{ (and nowhere else between 0 and 5).}$$

5 pts

(b) What is $F(2)$?

Solution: Since $F(2)$ is the area between $x = 0$ and $x = 2$, by the previous we have $\boxed{F(2) = 0}$.

5 pts

(c) What is $F'(4)$?

Solution: By the Fundamental Theorem of Calculus, we have $F'(4) = g(4) = \boxed{1}$.

5 pts

(d) What is $F''(1)$?

Solution: Since $F'(x) = g(x)$, we have $F''(x) = g'(x)$ (whenever $g'(x)$ is defined). The part of the graph of $g(x)$ near $x = 1$ is a line with slope 1, so

$$F''(1) = g'(1) = \boxed{1}\,.$$

5. The limit

$$\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \ln\left(3 + \frac{2k}{n}\right)$$

corresponds to a definite integral.

10 pts (a) State a definite integral that the limit corresponds to. To get full credit, you must give some explanation of how this integral and the limit are related. Do not calculate the integral.

Solution: Recall $\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(a + x_k)$, where $\Delta x = (b-a)/n$ and $x_k = k\Delta x$.
There are many possible correct answers.
Each term in the sum is of the form $\Delta x \cdot f(a + k\Delta x)$. If we take $a = 0$, this simplifies the identification process – we just need to figure out Δx and the rest is automatic.
The obvious choice is $\Delta x = 2/n$, giving $b = 2$ and $f(x) = \ln(3 + x)$, so our integral is $\boxed{\int_0^2 \ln(3 + x)\,dx}$.
Alternatively, you might take $a = 3$ and $\Delta x = 2/n$. In this case, things are pretty straightforward, getting $f(x) = \ln(x)$. Here we would get $\boxed{\int_3^5 \ln(x)\,dx}$.
Any integral equivalent to any of the above by a substitution will correspond to the same Riemann sum.

5 pts (b) Is the sum $\frac{2}{10} \sum_{k=1}^{10} \ln\left(3 + \frac{2k}{10}\right)$ larger or smaller than the integral above? Why? (You should not need to calculate either the integral or the sum in order to answer this question).

Solution: Since $\ln(x)$ is an increasing function, the right endpoint of each rectangle will be larger than the left, so the top of each rectangle is always above the graph. Hence, **the right-hand sum is always larger than the integral for an increasing function** as in this case.