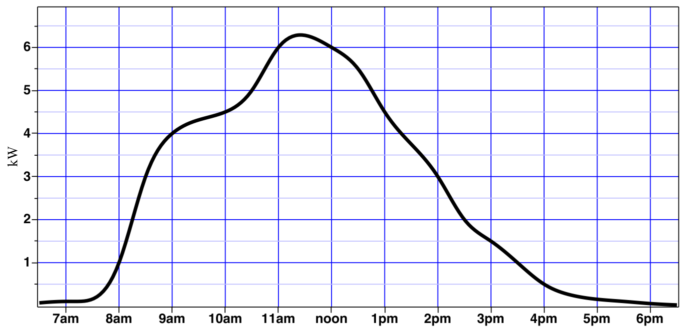
MAT 126 Solutions to Midterm 1 (TVC15)

20 pts

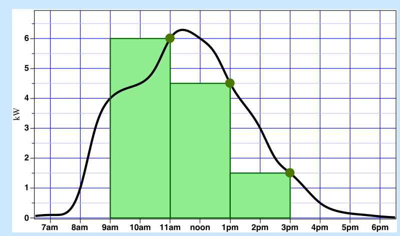
1. At right is shown a graph of the amount of electricity generated (in kilowatts) by a home solar array on a recent day, as a function of time. Let $E(t)$ denote this function. Then the total power produced (in kilowatt-hours) between times $t = a$ and $t = b$ is given by the integral $\int_a^b E(t)\,dt$.



(a) Use a Riemann sum with three intervals of equal width evaluated at **the right end-point**, to calculate the total power generated between 9am and 3pm.

Solution:
Using three rectangles evaluated on the right, since the timespan is 6 hours, the width of each is 2h. This gives

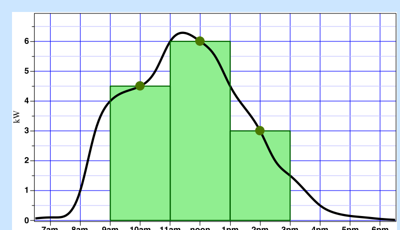
$$2 \cdot (6 + 4.5 + 1.5) = 24 \text{ kilowatt-hours.}$$



(b) Use a Riemann sum with three intervals of equal width evaluated at **the midpoint**, to calculate the total power generated between 9am and 3pm.

Solution:
Using three rectangles evaluated at their midpoints where the width of each is 2h, we obtain

$$2 \cdot (4.5 + 6 + 3) = 27 \text{ kilowatt-hours.}$$



2. Evaluate each of the indefinite integrals below.

5 pts

(a) $\int e^{1-5t}\,dt$

Solution: Make the substitution $u = 1 - 5t$, so $du = -dt$. Thus we have

$$\int e^{1-5t}\,dt = -\int e^u\,du = -e^u + C = \boxed{C - e^{1-5t}}\ .$$

5 pts (b) $\int \theta \sin 4\theta \; d\theta$

Solution: Integrating by parts with $u = \theta$ and $dv = \sin 4\theta \; d\theta$, we obtain $du = d\theta$ and $v = \int dv = -\cos 4\theta/4$. This yields

$$\int \theta \sin 4\theta \; d\theta = -\frac{\theta \cos 4\theta}{4} + \frac{1}{4}\int \cos 4\theta \; d\theta = \boxed{C - \frac{\theta \cos 4\theta}{4} + \frac{\sin 4\theta}{16}}.$$

3. Evaluate each of the definite integrals below.

5 pts (a) $\displaystyle\int_0^{\sqrt{3}} \frac{3x+1}{1+x^2} \, dx$

Solution:

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{3x+1}{1+x^2} \, dx &= \int_0^{\sqrt{3}} \frac{3x \; dx}{1+x^2} + \int_0^{\sqrt{3}} \frac{dx}{1+x^2} \\ &= 3\int_1^4 \frac{du/2}{u} + \arctan(x)\Big|_0^{\sqrt{3}} = \tfrac{3}{2}\ln|u|\Big|_1^4 + \left(\frac{\pi}{3} - 0\right) = \boxed{\tfrac{3}{2}\ln|4| + \frac{\pi}{3}},\end{aligned}$$

with the substitution $u = 1 + x^2$, $du = 2x \; dx$ in the first integral after pulling out the 3.

5 pts (b) $\displaystyle\int_{-1}^{2} \left|w^3\right| \; dw$

Solution: Recall that $|w^3| = \begin{cases} -w^3 & \text{if } w < 0 \\ w^3 & \text{for } w \geq 0 \end{cases}$. Split the integral at zero to obtain

$$\begin{aligned}\int_{-1}^{2} \left|w^3\right| \; dw = -\int_{-1}^{0} w^3 \, dw + \int_0^2 w^3 \, dw &= -\frac{w^4}{4}\Big|_{-1}^{0} + \frac{w^4}{4}\Big|_0^2 \\ &= -\left(0 - \frac{1}{4}\right) + \left(\frac{2^4}{4} - 0\right) = \boxed{\frac{1+2^4}{4} = \frac{17}{4}}.\end{aligned}$$

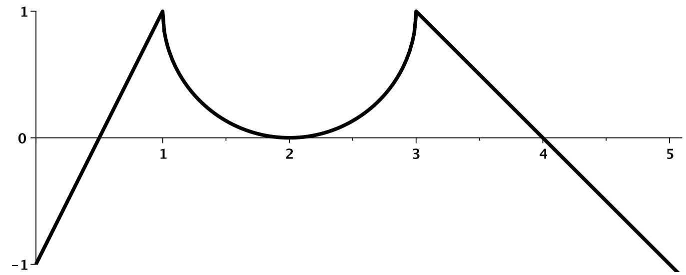
5 pts (c) $\displaystyle\int_1^3 \ln(1+z) \; dz$

Solution: It isn't strictly necessary, but let's make the substitution $x = 1 + z$ (so $dx = dz$) and the integral becomes $\displaystyle\int_2^4 \ln(x) \; dx$. Then integrate by parts with $u = \ln(x)$ and $dv = dx$. This gives us $du = dx/x$ and $v = x$ (so $v \, du = dx$).

$$\int_2^4 \ln(x) \; dx = x\ln(x)\Big|_2^4 - \int_2^4 dx = (4\ln 4 - 2\ln 2) - (4 - 2) = \boxed{4\ln 4 - 2\ln 2 - 2}.$$

4. Let $g(t)$ be the function given by

$$g(t) = \begin{cases} 2t-1 & t \le 1 \\ 1-\sqrt{1-(t-2)^2} & 1 < t \le 3 \\ 4-t & t > 3 \end{cases}$$



with its graph shown at right. Let

$$F(x) = \int_0^x g(t)\,dt\,.$$

A typo on the exam incorrectly listed $g(t) = 2 - 2t$ for $t > 3$, although the graph used $4 - t$. If you used that for g, you should receive full credit.

5 pts (a) For what values of x between 0 and 5 is $F(x) \le 0$?

Solution: Observe that the (signed) area between the graph and the axis for $0 < x < \frac{1}{2}$ is a triangle with base $\frac{1}{2}$, height -1, so this area is $-\frac{1}{4}$. Similarly, the area for x between $\frac{1}{2}$ and $\frac{2}{1}$ is $-\frac{1}{4}$, so $F(1) = 0$, and for all $x < 1$, $F(x)$ is negative.

Notice that $g(t) > 0$ for $1 < x < 4$, and by an argument similar to the above, the total area for $x \in [4, 5]$ is 0. This means $F(x)$ is positive for $x \in [1, 5]$, and so

$$\boxed{F(x) \le 0 \text{ for } 0 \le x \le 1} \text{ (and nowhere else between 0 and 5).}$$

5 pts (b) What is $F(1)$?

Solution: Since $F(1)$ is the area between $x = 0$ and $x = 1$, by the previous we have $\boxed{F(1) = 0}$.

5 pts (c) What is $F'(1)$?

Solution: By the Fundamental Theorem of Calculus, we have $F'(1) = g(1) = \boxed{1}$.

5 pts (d) What is $F''(4)$?

Solution: Since $F'(x) = g(x)$, we have $F''(x) = g'(x)$ (whenever $g'(x)$ is defined). The part of the graph of $g(x)$ near $x = 4$ is a line with slope -1, so

$$F''(4) = g'(4) = \boxed{-1}\,.$$

Some students used the formula with a typo to answer that $F''(4) = -2$. Such an answer should receive full credit.

5. The limit

$$\lim_{n\to\infty} \frac{4}{n} \sum_{k=1}^{n} \ln\left(1 + \frac{8k}{n}\right)$$

corresponds to a definite integral.

10 pts (a) State a definite integral that the limit corresponds to. To get full credit, you must give some explanation of how this integral and the limit are related. Do not calculate the integral.

Solution: Recall $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(a + x_k)$, where $\Delta x = (b-a)/n$ and $x_k = k\Delta x$.
There are many possible correct answers.
Each term in the sum is of the form $\Delta x \cdot f(a + k\Delta x)$. If we take $a = 0$, this simplifies the identification process – we just need to figure out Δx and the rest is automatic.
We now need to choose between $\Delta x = 4/n$ and $\Delta x = 8/n$.
If we take $\Delta x = 8/n$, we get $b = 8$ but we will have to multiply the factor of $2/n$ by two to make things match. This means we must divide the whole result by two in order to compensate. This gives $\boxed{\frac{1}{2}\int_0^8 \ln(1+x)\,dx}$.
Or we could take $\Delta x = 4/n$ (and still $a = 0$). Then we will have $b = 4$, and $x_k = 1 + 2k\Delta x$, giving use $f(x) = \ln(1+2x)$ and the integral $\boxed{\int_0^4 \ln(1+2x)\,dx}$.
Alternatively, you might take $a = 1$ and $\Delta x = 8/n$. In this case, things are pretty straightforward, getting $f(x) = \ln(x)$, but we have to adjust by the factor of $1/2$. Here we would get $\boxed{\frac{1}{2}\int_1^9 \ln(x)\,dx}$.
If you take $a \neq 0$ and $\Delta x = 4/n$, then things get more complicated, because you have to squeeze $1 + 2k\Delta x$ into the form $x_i = a + k\Delta x$ to arrive at $f(x)$. There are many ways to do this.
Any integral equivalent to any of the above by a substitution will correspond to the same Riemann sum.

5 pts (b) Is the sum $\frac{4}{10}\sum_{k=1}^{10}\ln\left(1+\frac{8k}{10}\right)$ larger or smaller than the integral above? Why? (You should not need to calculate either the integral or the sum in order to answer this question).

Solution: Since $\ln(x)$ is an increasing function, the right endpoint of each rectangle will be larger than the left, so the top of each rectangle is always above the graph. Hence, **the right-hand sum is always larger than the integral for an increasing function** as in this case.