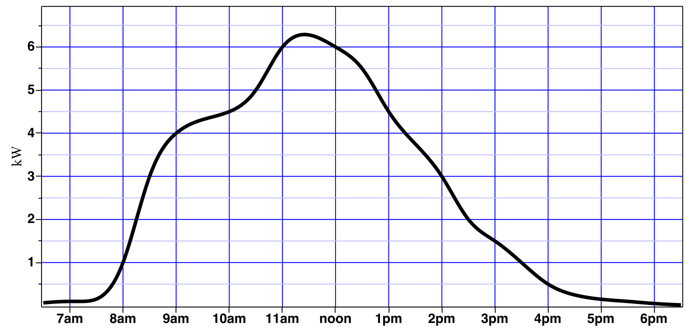
# MATH 126 Solutions to Midterm 1 (Ziggy)

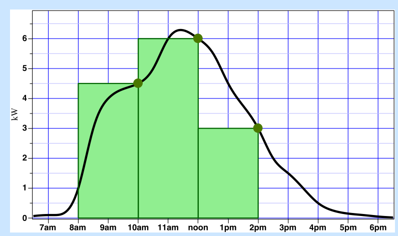
20 pts 1. At right is shown a graph of the amount of electricity generated (in kilowatts) by a home solar array on a recent day, as a function of time. Let $E(t)$ denote this function. Then the total power produced (in kilowatt-hours) between times $t = a$ and $t = b$ is given by the integral $\int_a^b E(t)\,dt$.




(a) Use a Riemann sum with three intervals of equal width evaluated at **the right endpoint**, to calculate the total power generated between 8am and 2pm.

**Solution:**
Using three rectangles evaluated on the right, since the timespan is 6 hours, the width of each is 2h. This gives

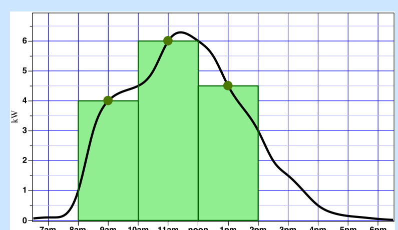
$$2 \cdot (4.5 + 6 + 3) = 27 \text{ kilowatt-hours.}$$



(b) Use a Riemann sum with three intervals of equal width evaluated at **the midpoint**, to calculate the total power generated between 8am and 2pm.

**Solution:**
Using three rectangles evaluated at their midpoints where the width of each is 2h, we obtain

$$2 \cdot (4 + 6 + 4.5) = 29 \text{ kilowatt-hours.}$$



2. Evaluate each of the indefinite integrals below.

5 pts (a) $\int e^{1-2t}\,dt$

**Solution:** Make the substitution $u = 1 - 2t$, so $du = -dt$. Thus we have

$$\int e^{1-2t}\,dt = -\int e^u\,du = -e^u + C = \boxed{C - e^{1-2t}}\ .$$

5 pts (b) $\int \theta \sin 2\theta \, d\theta$

**Solution:** Integrating by parts with $u = \theta$ and $dv = \sin 2\theta \ d\theta$, we obtain $du = d\theta$ and $v = \int dv = -\cos 2\theta/2$. This yields

$$\int \theta \sin 2\theta \, d\theta = -\frac{\theta \cos 2\theta}{2} + \frac{1}{2}\int \cos 2\theta \, d\theta = \boxed{C - \frac{\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4}}.$$

3. Evaluate each of the definite integrals below.

5 pts (a) $\int_0^{\sqrt{3}} \frac{5x+1}{1+x^2} \, dx$

**Solution:**

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{5x+1}{1+x^2} \, dx &= \int_0^{\sqrt{3}} \frac{5x \, dx}{1+x^2} + \int_0^{\sqrt{3}} \frac{dx}{1+x^2} \\ &= 5\int_1^4 \frac{du/2}{u} + \arctan(x)\Big|_0^{\sqrt{3}} = \tfrac{5}{2}\ln|u|\Big|_1^4 + \left(\frac{\pi}{3} - 0\right) = \boxed{\tfrac{5}{2}\ln|4| + \frac{\pi}{3}},\end{aligned}$$

with the substitution $u = 1 + x^2$, $du = 2x \, dx$ in the first integral after pulling out the $5$.

5 pts (b) $\int_{-1}^{3} \left|w^3\right| \, dw$

**Solution:** Recall that $|w^3| = \begin{cases} -w^3 & \text{if } w < 0 \\ w^3 & \text{for } w \geq 0 \end{cases}$. Split the integral at zero to obtain

$$\begin{aligned}\int_{-1}^{3} \left|w^3\right| \, dw = -\int_{-1}^{0} w^3 \, dw + \int_0^3 w^3 \, dw = -\frac{w^4}{4}\Big|_{-1}^{0} + \frac{w^4}{4}\Big|_0^3 \\ = -\left(0 - \frac{1}{4}\right) + \left(\frac{3^4}{4} - 0\right) = \boxed{\frac{1+3^4}{4} = \frac{82}{4} = \frac{41}{2}}.\end{aligned}$$

5 pts (c) $\int_1^5 \ln(1+z) \, dz$

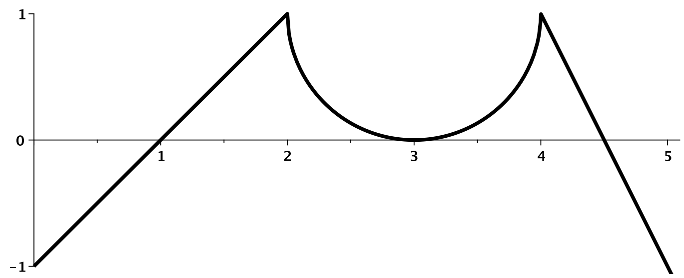
**Solution:** It isn't strictly necessary, but let's make the substitution $x = 1 + z$ (so $dx = dz$) and the integral becomes $\int_2^6 \ln(x) \, dx$. Then integrate by parts with $u = \ln(x)$ and $dv = dx$. This gives us $du = dx/x$ and $v = x$ (so $v \, du = dx$).

$$\int_2^6 \ln(x) \, dx = x\ln(x)\Big|_2^6 - \int_2^6 dx = (6\ln 6 - 2\ln 2) - (6-2) = \boxed{6\ln 6 - 2\ln 2 - 4}.$$

4. Let $g(t)$ be the function given by

$$g(t) = \begin{cases} t-1 & t \leq 2 \\ 1-\sqrt{1-(t-3)^2} & 2 < t \leq 4 \\ 9-2t & t > 4 \end{cases}$$

with its graph shown at right. Let




$$F(x) = \int_0^x g(t)\,dt\,.$$

$\boxed{\text{5 pts}}$ (a) For what values of $x$ between $0$ and $5$ is $F(x) \leq 0$?

**Solution:** Observe that the (signed) area between the graph and the axis for $0 < x < 1$ is a triangle with base $1$, height $-1$, so this area is $-\frac{1}{2}$. Similarly, the area for $x$ between $1$ and $1$ is $-\frac{1}{2}$, so $F(2) = 0$, and for all $x < 2$, $F(x)$ is negative.
Notice that $g(t) > 0$ for $2 < x < 4.5$, and by an argument similar to the above, the total area for $x \in [3, 5]$ is 0. This means $F(x)$ is positive for $x \in [2, 5]$, and so

$$\boxed{F(x) \leq 0 \text{ for } 0 \leq x \leq 2}$$ (and nowhere else between 0 and 5).

$\boxed{\text{5 pts}}$ (b) What is $F(1)$?

**Solution:** Since $F(1)$ is the area between $x = 0$ and $x = 1$, by the previous we have $\boxed{F(1) = -\frac{1}{2}}$.

$\boxed{\text{5 pts}}$ (c) What is $F'(2)$?

**Solution:** By the Fundamental Theorem of Calculus, we have $F'(2) = g(2) = \boxed{1}$.

$\boxed{\text{5 pts}}$ (d) What is $F''(1)$?

**Solution:** Since $F'(x) = g(x)$, we have $F''(x) = g'(x)$ (whenever $g'(x)$ is defined). The part of the graph of $g(x)$ near $x = 1$ is a line with slope $1$, so

$$F''(1) = g'(1) = \boxed{1}\,.$$

5. The limit
$$\lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \ln\left(2 + \frac{6k}{n}\right)$$
corresponds to a definite integral.

10 pts (a) State a definite integral that the limit corresponds to. To get full credit, you must give some explanation of how this integral and the limit are related. Do not calculate the integral.

**Solution:** Recall $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(a + x_k)$, where $\Delta x = (b-a)/n$ and $x_k = k\Delta x$.
There are many possible correct answers.
Each term in the sum is of the form $\Delta x \cdot f(a + k\Delta x)$. If we take $a = 0$, this simplifies the identification process – we just need to figure out $\Delta x$ and the rest is automatic.
We now need to choose between $\Delta x = 3/n$ and $\Delta x = 6/n$.
If we take $\Delta x = 6/n$, we get $b = 6$ but we will have to multiply the factor of $2/n$ by two to make things match. This means we must divide the whole result by two in order to compensate. This gives $\boxed{\displaystyle\frac{1}{2}\int_0^6 \ln(2+x)\,dx}$.
Or we could take $\Delta x = 3/n$ (and still $a = 0$). Then we will have $b = 3$, and $x_k = 2 + 2k\Delta x$, giving use $f(x) = \ln(2+2x)$ and the integral $\boxed{\displaystyle\int_0^3 \ln(2+2x)\,dx}$.
Alternatively, you might take $a = 2$ and $\Delta x = 6/n$. In this case, things are pretty straightforward, getting $f(x) = \ln(x)$, but we have to adjust by the factor of $1/2$. Here we would get $\boxed{\displaystyle\frac{1}{2}\int_2^8 \ln(x)\,dx}$.
If you take $a \neq 0$ and $\Delta x = 3/n$, then things get more complicated, because you have to squeeze $2 + 2k\Delta x$ into the form $x_i = a + k\Delta x$ to arrive at $f(x)$. There are many ways to do this.
Any integral equivalent to any of the above by a substitution will correspond to the same Riemann sum.

5 pts (b) Is the sum $\dfrac{3}{10}\displaystyle\sum_{k=1}^{10} \ln\left(2+\dfrac{6k}{10}\right)$ larger or smaller than the integral above? Why? (You should not need to calculate either the integral or the sum in order to answer this question).

**Solution:** Since $\ln(x)$ is an increasing function, the right endpoint of each rectangle will be larger than the left, so the top of each rectangle is always above the graph. Hence, **the right-hand sum is always larger than the integral for an increasing function** as in this case.