MATH 126 Solutions to Midterm 1 (Bird)

20 pts

1. A passenger plane touches down at Laguardia with a ground speed of 200 feet per second (about 136 mph). The pilot then reverses the engines to slow the plane down to 30 ft/sec, at which speed she can safely taxi to the gate. The table below gives the speed at t seconds after touchdown.

time (seconds)	0	2	4	6	8
speed (ft/sec)	200	160	100	40	30

Using the information above and assuming the plane's speed is decreasing continuously, use a Riemann sum to calculate both an upper bound and a lower bound on the distance the plane traveled (in feet) during the 8 seconds after touchdown.

For full credit, you must indicate clearly how you arrived at each answer.

Solution: Let $v(t)$ be the velocity at the tth second. Since $v(t)$ is decreasing, the upper bound will be given by the left sum, and the lower bound by the right sum.

Here we have the width of each rectangle as 2, and we use 4 rectangles (since 5 points define 4 rectangles). The right sum is

$$2(v(2) + v(4) + v(6) + v(8)) = 2(160 + 100 + 40 + 30) = 660$$

(which is a lower bound since the speed is decreasing), and the left sum is

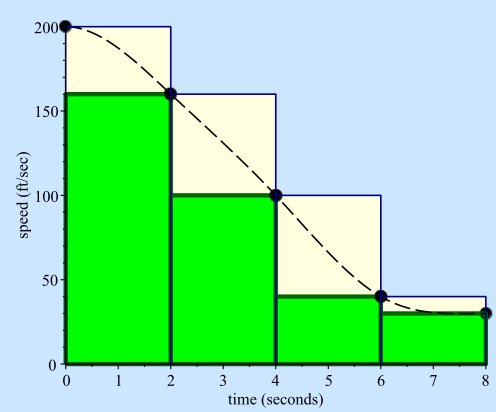
$$2(v(0) + v(2) + v(4) + v(6)) = 2(200 + 160 + 100 + 40) = 1000.$$

This means that

$$\boxed{660 \text{ feet}} \le \text{distance traveled} \le \boxed{1000 \text{ feet}}.$$

If you want to visualize this process graphically, see the image at right. This is, of course, not necessary, although some people drew a similar graph anyway.

The dashed curve represents a *guess* at $v(t)$ (the speed of the airplane), since we only know the speeds at the times in the chart.



2. Evaluate each of the integrals below.

8 pts (a) $\displaystyle\int_0^1 \frac{5x+1}{1+x^2}\,dx$

Solution:

$$\begin{aligned}\int_0^1 \frac{5x+1}{1+x^2}\,dx &= \int_0^1 \frac{5x\,dx}{1+x^2} + \int_0^1 \frac{dx}{1+x^2}\\ &= 5\int_1^2 \frac{du/2}{u} + \arctan(x)\Big|_0^1 = \tfrac{5}{2}\ln|u|\Big|_1^2 + \Big(\frac{\pi}{4} - 0\Big) = \boxed{\tfrac{5}{2}\ln|2| + \frac{\pi}{4}},\end{aligned}$$

where we made the substitution $u = 1+x^2$, $du = 2x\,dx$ in the first integral.

8 pts (b) $\displaystyle\int x\sqrt{x-1}\,dx$

Solution: Make the substitution $w = x-1$ so that $dw = dx$ and $x = w+1$ to get

$$\int (w+1)w^{1/2}\,dx = \int w^{3/2} + w^{1/2}\,dw = \tfrac{2}{5}w^{5/2} + \tfrac{2}{3}w^{3/2} + C = \boxed{\tfrac{2}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + C}.$$

You could also do this integral by parts, taking $u = x$ and $dv = \sqrt{x-1}\,dx$.
Then $du = dx$ and $v = \frac{2}{3}(x-1)^{3/2}$, so we have

$$\int x\sqrt{x-1}\,dx = \tfrac{2}{3}x(x-1)^{3/2} - \tfrac{2}{3}\int (x-1)^{3/2}dx = \boxed{\tfrac{2}{3}x(x-1)^{3/2} - \tfrac{4}{15}(x-1)^{5/2} + C}.$$

A little algebra shows these apparently different answers are, in fact, equal.

8 pts (c) $\displaystyle\int_0^{2\pi} |\sin x|\,dx$

Solution: First, notice that $\sin x \le 0$ for $\pi \le x \le 2\pi$, so $|\sin x| = \sin x$ for $0 \le x \le \pi$ but $|\sin x| = -\sin x$ when $\pi \le x \le 2\pi$. This means we have

$$\begin{aligned}\int_0^{2\pi} |\sin x|\,dx &= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi}\\ &= \big(-\cos(\pi) + \cos(0)\big) + \big(\cos(2\pi) - \cos(\pi)\big) = (1+1) + (1+1) = \boxed{4}.\end{aligned}$$

8 pts (d) $\displaystyle\int xe^{2x}\,dx$

Solution: Here we integrate by parts. Take $u = x$ and $dv = e^{2x}dx$,
so $du = dx$ and $v = \displaystyle\int e^{2x}dx = \frac{e^{2x}}{2}$. Then we have

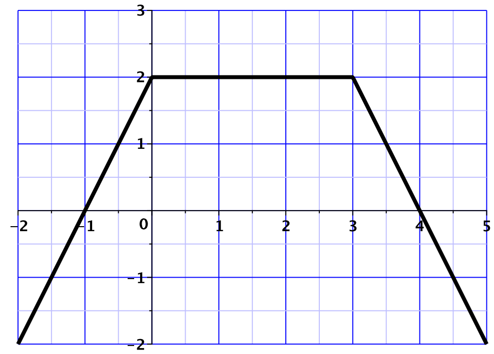
$$\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx = \boxed{\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C}.$$

3. Let $g(t)$ be the function with graph shown at right, and let

$$F(x) = \int_{-2}^{x} g(t)\,dt$$

for $-2 \le x \le 5$.

For full credit, give at least a little justification for each of your answers to the questions below.



5 pts (a) On what interval(s) is $F(x) \le 0$?

If there are none, write "None".

Solution: Observe that the (signed) area between the graph and the axis for $-2 < x < 1$ is a triangle with base 1, height -2, so this area is -2. Similarly, the area for x between 1 and 0 is $+2$. Thus, $F(x)$ is negative in this region. However, for $0 < x < 5$, we have $g(t) > 0$ and there is a total area of $+7$ for $0 < x < 5$, hence $F(x)$ is positive here. Finally, while the area for x between 4 and 5 is -1, $F(x)$ remains positive (since $F(4) = 7$ and $F(5) = 6$). Thus

$$F(x) \le 0 \text{ for } \boxed{-2 \le x \le 0} \text{ (and nowhere else).}$$

5 pts (b) What is $F(3)$?

If $F(3)$ is not defined, write "DNE".

Solution: Since $F(0) = 0$ (see above), we just need to count the boxes in the rectangle with base between $x = 0$ and $x = 3$ to obtain

$$F(3) = \boxed{6}.$$

5 pts (c) What is $F'(0)$?

If $F'(0)$ does not exist, write "DNE".

Solution: By the Fundamental Theorem of Calculus, we have $F'(0) = g(0) = \boxed{2}$.

5 pts (d) What is $F''(4)$?

If $F''(4)$ does not exist, write "DNE".

Solution: Since $F'(x) = g(x)$, we have $F''(x) = g'(x)$. The part of the graph of $g(x)$ between $x = 3$ and $x = 5$ is a line with slope -2, so

$$F''(4) = g'(4) = \boxed{-2}.$$

4. Let $h(x) = x^2 + 2x + 2$.

8 pts (a) Use a Riemann sum with three rectangles, evaluated on the right-hand side, to approximate $\int_{-4}^{2} h(x)\,dx$.

Solution: Since we have $a = -4$ and $b = 2$ and are using three rectangles, the width of each is $\Delta x = (2+4)/3 = 2$, and the relevant points are $x_0 = -4$, $x_1 = -2$, $x_2 = 0$, and $x_3 = 2$. Evaluating on the right, we have

$$2\big(h(-2) + h(0) + h(2)\big) = 2(2+2+10) = \boxed{28}.$$

8 pts (b) Express $\int_{-4}^{2} x^2 + 2x + 2\,dx$ as a limit of a Riemann sum (with n rectangles).
Your final answer should not include symbols like Δx or x_i.

Solution: Recall that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = (b-a)/n$ and $x_i = a + \Delta x$.
In this case, we have $\Delta x = 6/n$ and so $x_i = \frac{6i}{n} - 4$, giving

$$\int_{-4}^{2} h(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} h\left(\tfrac{6i}{n} - 4\right)\frac{6}{n} = \boxed{\lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(\left(\tfrac{6i}{n} - 4\right)^2 + 2\left(\tfrac{6i}{n} - 4\right) + 2\right)}.$$

8 pts (c) Calculate $\int_{-4}^{2} x^2 + 2x + 2\,dx$ exactly.

Solution: (Unfortunately, there was a typo on the exam and $\int_{-4}^{2} x^2 + 3x + 2\,dx$ was asked for. Some people fixed it, some didn't.)

$$\begin{aligned}\int_{-4}^{2} x^2 + 2x + 2\,dx &= \frac{x^3}{3} + x^2 + 2x\Big|_{-4}^{2} \\ &= \left(\frac{8}{3} + 4 + 4\right) - \left(\frac{-64}{3} + 16 - 8\right) = \frac{72}{3} + 0 = \boxed{24}.\end{aligned}$$

If you didn't fix the typo, the following solution is also correct:

$$\begin{aligned}\int_{-4}^{2} x^2 + 3x + 2\,dx &= \frac{x^3}{3} + \frac{3x^2}{2} + 2x\Big|_{-4}^{2} \\ &= \left(\frac{8}{3} + 6 + 4\right) - \left(\frac{-64}{3} + 24 - 8\right) = \frac{72}{3} - 6 = \boxed{18}.\end{aligned}$$

$\boxed{\text{15 pts}}$ 5. Calculate the indefinite integral $\int e^x \sin(4x)\, dx$.

Solution: Integrating by parts with $u = \sin(4x)$ and $dv = e^x dx$,
we have $du = 4\cos(4x)dx$ and $v = e^x$, giving

$$\int e^x \sin(4x)\, dx = e^x \sin(4x) - 4\int e^x \cos(4x)\, dx.$$

Integrate by parts again (with $u = \cos(4x)$ and $dv = e^x dx$, so $du = -4\sin(4x)dx$ and $v = e^x$) to obtain

$$\begin{aligned}\int e^x \sin(4x)\, dx &= e^x \sin(4x) - 4\Big(e^x \cos(4x) + 4\int e^x \sin(4x)\, dx\Big)\\ &= e^x \sin(4x) - 4e^x \cos(4x) - 16\int e^x \sin(4x)\, dx.\end{aligned}$$

Adding $16\int e^x \sin(4x)\, dx$ to both sides gives

$$17\int e^x \sin(4x)\, dx = e^x \sin(4x) - 4e^x \cos(4x)$$

and so

$$\boxed{\int e^x \sin(4x)\, dx = \frac{e^x \sin(4x) - 4e^x \cos(4x)}{17} + C}.$$

Note that you could also do this by parts taking $u = e^x$ and $dv = \sin(4x)$. The constants are a bit messier, but it works about the same way.

We have $du = e^x dx$ and $v = -\frac{1}{4}\cos(4x)$, giving

$$\int e^x \sin(4x)\, dx = -\frac{e^x}{4}\cos(4x) + \tfrac{1}{4}\int e^x \cos(4x)\, dx.$$

Integrate by parts again (with $u = e^x$ and $dv = \cos(4x)dx$, so $du = e^x dx$ and $v = \frac{1}{4}\sin(4x)$) to obtain

$$\begin{aligned}\int e^x \sin(4x)\, dx &= -\frac{e^x}{4}\cos(4x) + \frac{1}{4}\Big(\frac{e^x}{4}\sin(4x) - \tfrac{1}{4}\int e^x \sin(4x)\, dx\Big)\\ &= -\frac{e^x}{4}\cos(4x) + \frac{e^x}{16}\sin(4x) - \tfrac{1}{16}\int e^x \sin(4x)\, dx.\end{aligned}$$

Adding $\frac{1}{16}\int e^x \sin(4x)\, dx$ to both sides gives

$$\tfrac{17}{16}\int e^x \sin(4x)\, dx = \frac{e^x}{16}\sin(4x) - \frac{e^x}{4}\cos(4x) = \frac{e^x \sin(4x) - 4e^x \cos(4x)}{16}$$

and then multiplying both sides by $\frac{16}{17}$ gives the same answer as above, namely

$$\boxed{\int e^x \sin(4x)\, dx = \frac{e^x \sin(4x) - 4e^x \cos(4x)}{17} + C}.$$