1) $\int \cos^2 x\,dx = \frac{1}{2}\int \left(1+\cos 2x\right)dx = \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+C$

2) $\int \cos^2 x \sin^2 x dx = \int \left[\frac{1}{2}\left(1+\cos 2x\right)\frac{1}{2}\left(1-\cos 2x\right)\right]dx$

$= \frac{1}{4}\int \left(1-\cos^2 2x\right)dx = \frac{1}{4}\int \sin^2 xdx$

$= \frac{1}{8}\int \left(1-\cos 4x\right)dx = \frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+C$

3) $\int \cos^4 x\,dx = \int \left(\cos^2 x\right)^2 dx = \int \left[\frac{1}{2}\left(1+\cos 2x\right)\right]^2 dx$

$= \frac{1}{4}\int \left(1+2\cos 2x+\cos^2 2x\right)dx = \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2xdx + \frac{1}{4}\int \cos^2 2xdx$

$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2xdx + \frac{1}{4}\int \frac{1}{2}\left(1+\cos 4x\right)dx$

$= \frac{1}{4}x + \frac{1}{2}\frac{\sin 2x}{2} + \frac{1}{8}\left(x+\frac{\sin 4x}{4}\right)+C$

4) $\int \sin^4 x \cos x\,dx =$

Let $u=\sin x$ and $du=\cos xdx$. Then

$\int \sin^4 x\cos x\,dx = \int u^4 du = \frac{u^5}{5}+C = \frac{\sin^5 x}{5}+C$

5) $\int \cos^7 x \sin^3 x\,dx = \int \cos^7 x \sin^2 x \sin x\,dx$

$= \int \cos^7 x\left(1-\cos^2 x\right)\sin x\,dx$

Let $u=\cos x$ and $du=-\sin xdx$. Then

$= -\int u^7\left(1-u^2\right)du = -\int u^7 - u^9 du = -\frac{u^8}{8}+\frac{u^{10}}{10}+C = -\frac{\cos^8 x}{8}+\frac{\cos^{10} x}{10}+C$

6) $\int \tan^2 x\,dx = \int \sec^2 x - 1dx = \tan x - x + C$

7) $\int \tan^2 x \sec^2 x \, dx$

Let $u = \tan x$ and $du = \sec^2 x dx$.

Then $\int \tan^2 x \sec^2 x dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\tan^3 x}{3} + C$

8) $\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^2 x \left(1 + \tan^2 x\right) \sec^2 x \, dx$

Let $u = \tan x$ and $du = \sec^2 x dx$.

Then $\int \tan^2 x \left(1 + \tan^2 x\right) \sec^2 x \, dx = \int u^2 \left(1 + u^2\right) du = \int u^2 + u^4 du$

$= \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$

9) $\int \frac{8x - 7}{x^2 - x - 2} dx =$

$\frac{A}{x-2} + \frac{B}{x+1} = \frac{8x-7}{(x-2)(x+1)}$

$A(x+1) + B(x-2) = 8x - 7$

$Ax + A + Bx - 2B = 8x - 7$

$(A+B)x + (A-2B) = 8x - 7$

So $\begin{matrix} A + B = 8 \\ A - 2B = -7 \end{matrix}$ and $A = 3$, $B = 5$

Thus we can rewrite $\int \frac{8x-7}{x^2 - x - 2} dx = \int \frac{3}{x-2} dx + \int \frac{5}{x+1} dx$

$= 3\ln|x-2| + 5\ln|x+1| + C$

10) $\int \frac{4x-60}{x^2-15x+50}\,dx =$

$$\frac{A}{x-5}+\frac{B}{x-10}=\frac{4x-60}{(x-5)(x-10)}$$

$$A(x-10)+B(x-5)=4x-60$$

$$Ax-10A+Bx-5B=4x-60$$

$$(A+B)x+(-10A-5B)=4x-60$$

So $\begin{matrix} A+B=4 \\ -10A-5B=60 \end{matrix}$ and $A=8$, $B=-4$

Thus we can rewrite $\int \frac{4x-60}{x^2-15x+50}\,dx = \int \frac{8}{x-5}\,dx - \int \frac{4}{x-10}\,dx$

$$=8\ln|x-5|-4\ln|x-10|+C$$

11) $\int \frac{-2x^2-22x-11}{x^3-8x^2-x+8}dx = \int \frac{-2x^2-22x-11}{x^2(x-8)-1(x-8)}dx$

$$= \int \frac{-2x^2-22x-11}{(x^2-1)(x-8)}dx = \int \frac{-2x^2-22x-11}{(x+1)(x-1)(x-8)}dx$$

$$\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-8}=\frac{-2x^2-22x-11}{(x-1)(x+1)(x-8)}$$

$$A(x+1)(x-8)+B(x-1)(x-8)+C(x-1)(x+1)=-2x^2-22x-11$$

$$A(x^2-7x-8)+B(x^2-9x+8)+C(x^2-1)=-2x^2-22x-11$$

$$(A+B+C)x^2+(-7A-9B)x+(-8A+8B-C)=-2x^2-22x-11$$

$$A+B+C=-2$$

$$-7A-9B=-22$$

$$-8A+8B-C=-11$$

(Hint: Let $C=-2-A-B$ and substitute into the third equation)

$A=\frac{5}{2}$, $B=\frac{1}{2}$, and $C=-5$

Thus we can rewrite

$$\int \frac{-2x^2-22x-11}{(x+1)(x-1)(x-8)}dx = \frac{5}{2}\int \frac{dx}{x-1}+\frac{1}{2}\int \frac{dx}{x+1}-5\int \frac{dx}{x-8}$$

$$=\frac{5}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|-5\ln|x-8|+C$$

12) $\int \frac{5x^2-7x-31}{(x^2+9)(x-4)}dx =$

$\frac{Ax+B}{x^2+9}+\frac{C}{x-4}=\frac{5x^2-7x-31}{(x^2+9)(x-4)}$

$(Ax+B)(x-4)+C(x^2+9)=5x^2-7x-31$

$Ax^2-4Ax+Bx-4B+Cx^2+9C=5x^2-7x-31$

$(A+C)x^2+(-4A+B)x+(-4B+9C)=5x^2-7x-31$

$A+C=5$

$-4A+B=-7$

$-4B+9C=-31$

(Hint: Let $C=5-A$ and substitute into the third equation)

$A=\frac{104}{25}$, $B=\frac{241}{25}$, and $C=\frac{21}{25}$

Thus we can rewrite

$$\int \frac{5x^2-7x-31}{(x^2+9)(x-4)}dx=\frac{104}{25}\int \frac{xdx}{x^2+9}+\frac{241}{25}\int \frac{dx}{x^2+9}+\frac{21}{25}\int \frac{dx}{x-4}$$

For the first integral, let $u=x^2+9$ and $du=2xdx$, so $\frac{1}{2}du=dx$. Then

$$\frac{104}{25}\int \frac{xdx}{x^2+9}=\frac{104}{25}\frac{1}{2}\int \frac{du}{u}=\frac{52}{25}\ln|u|+C=\frac{52}{25}\ln|x^2+9|+C$$

The second integral is $\frac{241}{25}\int \frac{dx}{x^2+9}=\frac{241}{25}\frac{1}{9}\text{Tan}^{-1}\left(\frac{x}{3}\right)=\frac{241}{225}\text{Tan}^{-1}\left(\frac{x}{3}\right)+C$

And the third integral is $\frac{21}{25}\int \frac{dx}{x-4}=\frac{21}{25}\ln|x-4|+C$

13) $\int_1^\infty \frac{dx}{x^{10}} = \lim_{a\to\infty}\int_1^a \frac{dx}{x^{10}} = \lim_{a\to\infty}\left(\frac{x^{-9}}{-9}\right)\Bigg|_1^a = \lim_{a\to\infty}\left(-\frac{1}{9x^9}\right)\Bigg|_1^a$

$\lim_{a\to\infty}\left(-\frac{1}{9a^9}+\frac{1}{9}\right)=\frac{1}{9}$

14) $\int_{10}^\infty \frac{dx}{(x-2)^4} = \lim_{b\to\infty}\int_{10}^b \frac{dx}{(x-2)^4} = \lim_{b\to\infty}\frac{(x-2)^{-3}}{-3}\Bigg|_{10}^b$

$= \lim_{b\to\infty}\left(-\frac{1}{3(x-2)^3}\right)\Bigg|_{10}^b = \lim_{b\to\infty}\left(-\frac{1}{3(b-2)^3}+\frac{1}{3(8)^3}\right)=\frac{1}{3(8)^3}$

15) $\int_{10}^\infty \frac{e^{-x}dx}{1-e^{-x}} = \lim_{c\to\infty}\int_{10}^c \frac{e^{-x}dx}{1-e^{-x}}$.

Let $u=1-e^{-x}$ and $du=e^{-x}dx$. Then $\int \frac{e^{-x}dx}{1-e^{-x}} = \int\frac{du}{u} = \ln|u| = \ln\left|1-e^{-x}\right|$

$\lim_{c\to\infty}\ln\left|1-e^{-x}\right|\Big|_{10}^c = \lim_{c\to\infty}\ln\left|1-e^{-c}\right| - \lim_{c\to\infty}\ln\left|1-e^{-10}\right| = \ln 1 - \ln\left|1-e^{-10}\right| = -\ln\left|1-e^{-10}\right|$

16) $\int_0^1 \frac{dx}{2x-1} = \lim_{d\to\frac{1}{2}^-}\int_0^d \frac{dx}{2x-1} + \lim_{d\to\frac{1}{2}^+}\int_5^1 \frac{dx}{2x-1}$

$= \lim_{d\to\frac{1}{2}^-}\frac{\ln|2x-1|}{2}\Bigg|_0^d + \lim_{d\to\frac{1}{2}^+}\frac{\ln|2x-1|}{2}\Bigg|_d^1$

$= \lim_{d\to\frac{1}{2}^-}\frac{\ln|2d-1|}{2} - \frac{\ln 1}{2} + \lim_{d\to\frac{1}{2}^+}\frac{\ln 1}{2} - \frac{\ln|2d-1|}{2}$, which diverges.

17) $\displaystyle\int_{-5}^{5} \frac{dx}{\sqrt[3]{1-x}} = \lim_{g\to 1^-} \int_{-5}^{g} \frac{dx}{\sqrt[3]{1-x}} + \lim_{g\to 1^+} \int_{g}^{5} \frac{dx}{\sqrt[3]{1-x}}$

$$= \lim_{g\to 1^-} \left. \frac{(1+x)^{\frac{2}{3}}}{\frac{2}{3}} \right|_{-5}^{g} + \lim_{g\to 1^+} \left. \frac{(1+x)^{\frac{2}{3}}}{\frac{2}{3}} \right|_{g}^{5}$$

$$= \lim_{g\to 1^-} \left(\frac{3(1+g)^{\frac{2}{3}}}{2} - \frac{3(-4)^{\frac{2}{3}}}{2} \right) + \lim_{g\to 1^+} \left(\frac{3(6)^{\frac{2}{3}}}{2} - \frac{3(1+g)^{\frac{2}{3}}}{2} \right) = \frac{3(6)^{\frac{2}{3}}}{2} - \frac{3(-4)^{\frac{2}{3}}}{2}$$

18) The region *R* in the first quadrant is bounded by $y = x^2 + x - 2$ and $y = 4$.
Sketch the region *R* and find its area.
Find the volume that results when *R* is revolved about the *x*-axis.

First, find the intersection of the two curves:

$x^2 + x - 2 = 4$

$x^2 + x - 6 = 0$

$x = -3$ and $x = 2$

Area is: $\displaystyle\int_{-3}^{2} 4 - (x^2 + x - 2)dx = \int_{-3}^{2} 6 - x^2 - xdx$

$$\int_{-3}^{2} 6 - x^2 - xdx = \left. \left(6x - \frac{x^3}{3} - \frac{x^2}{2} \right) \right|_{-3}^{2} = \frac{125}{6}$$

Volume is $\displaystyle\pi \int_{-3}^{2} (4)^2 - (x^2 + x - 2)^2 dx = \pi \int_{-3}^{2} 16 - (x^4 + 2x^3 - 3x^2 + 4)dx$

$$= \pi \int_{-3}^{2} 12 - x^4 - 2x^3 + 3x^2 dx = \pi \left. \left(12x - \frac{x^5}{5} + x^3 \right) \right|_{-3}^{2} = \frac{145\pi}{2}$$

19) The region *R* in the fourth quadrant is bounded by $y = x^3 - x^2 - 12x$ and $y = 0$ (the *x*-axis). Sketch the region *R* and find its area.
Find the volume that results when *R* is revolved about the *x*-axis.

Factor the curve to find where it crosses the *x*-axis:

$$x^3 - x^2 - 12x = x(x-4)(x+3)$$

Area: $\int_0^4 0 - \left(x^3 - x^2 - 12x\right)dx = \int_0^4 -x^3 + x^2 + 12x dx$

$$= \left(-\frac{x^4}{4} + \frac{x^3}{3} + 6x^2\right)\Bigg|_0^4 = \frac{160}{3}$$

Volume: $\pi\int_0^4 (0)^2 - \left(x^3 - x^2 - 12x\right)^2 dx = \pi\int_0^4 -x^6 + 2x^5 + 23x^4 - 24x^3 - 144x^2 dx$

$$= \pi\left(-\frac{x^7}{7} + \frac{x^6}{3} + \frac{23x^5}{5} - 6x^4 - 48x^3\right)\Bigg|_0^4 = \pi\frac{69632}{105}$$

20) The region *R* in the first quadrant is bounded by $y = \sin x$ and $y = x^2 - \pi x$ from $x = 0$ to $x = \frac{3\pi}{2}$.
Sketch the region *R* and find its area.

Area: $\int_0^{\pi} \sin x - \left(x^2 - \pi x\right)dx + \int_{\pi}^{\frac{3\pi}{2}} \left(x^2 - \pi x\right) - \sin x dx$

$$= \left(-\cos x - \frac{x^3}{3} + \frac{\pi x^2}{2}\right)\Bigg|_0^{\pi} + \left(\frac{x^3}{3} - \frac{\pi x^2}{2} + \cos x\right)\Bigg|_{\pi}^{\frac{3\pi}{2}}$$