1. A liquid leaked from a tank at a rate $r(t)$, where r is in liters per hour, and t is in hours. The rate decreased as time passed. This rate was measured every 2 hours, and the result is given below. Find an upper estimate for the total amount of oil that has leaked out after 6 hours.

t	0	2	4	6
r(t)	9	7.5	7	6

Solution: We are being asked to calculate a Riemann sum. Since the rate of leakage was decreasing, the left sum will give us an upper estimate. There are 6 hours involved, measured every 2 hours, we will have a sum with 3 rectangles, each of width 2.

Our estimate is

$$2r(0) + 2r(2) + 2r(4) = 2 \cdot 9 + 2 \cdot 7.5 + 2 \cdot 7 = 47$$

There should be no more than 47 liters of oil that leaked out during the 6 hours. (Note that we do NOT want to use $r(6)$ in our estimate, since that would tell us about what happened AFTER the 6th hour ended.

(Although it wasn't asked, we could do a right sum to get a lower bound on the amount of oil lost. In this case, we would get $2r(2) + 2r(4) + 2r(6) = 41$, so the total amount of oil that leaked in the given period was between 41 and 47 liters.)

2.

$$\int_2^3 f(x)dx = 1/3, \qquad \int_3^5 f(x)dx = 6, \qquad \int_2^5 g(x)dx = 4/5.$$

Find

$$\int_2^5 \big(g(x) + 4f(x) + 5\big)dx.$$

Solution: We rewrite this in terms of what we are given. We have

$$\int_2^5 (g(x) + 4f(x) + 5)dx = \int_2^5 g(x)\ dx + 4\int_2^5 f(x)\ dx + \int_2^5 5\ dx.$$

The first integral is given as 4/5, and the last is the area of a 5×3 rectangle, so it is 15. To do the middle integral, we use the fact that $\int_2^5 f(x)\ dx = \int_2^3 f(x)\ dx + \int_3^5 f(x)\ dx = 1/3 + 6$. This means we have

$$\int_2^5 \big(g(x) + 4f(x) + 5\big)dx = \frac{4}{5} + 4\left(6 + \frac{1}{3}\right) + 15 = \frac{617}{15}$$

Sorry about the fractions...

3. Express the integral as the limit of Riemann sums. Do not evaluate the limit. Do not use Δx or x_i^* in your final answer; instead, plug in the formulas for these.

$$\int_2^{12} \frac{\sqrt[3]{x}dx}{2+3x}$$

Solution: Let's write this as a right sum with n equal rectangles.

The width Δx of each rectangle will be $\dfrac{12-2}{n} = \dfrac{10}{n}$.

Since we are doing right sums, we divide the interval $[2, 10]$ into n pieces, and choose our x_i^* on the right (larger) side of each piece. Thus, we have $x_1^* = 2 + \frac{10}{n}$, $x_2^* = 2 + \frac{20}{n}$, $x_3^* = 2 + \frac{30}{n}$, and so on. More compactly, we can write this as

$$x_i^* = 2 + \frac{10i}{n} \qquad i \le i \le n.$$

This means that the area of the i^{th} rectangle will be

$$(\Delta x) f\left(x_i^*\right) = \left(\frac{10}{n}\right) f\left(2 + \frac{10i}{n}\right) = \left(\frac{10}{n}\right) \frac{\sqrt[3]{2+\frac{10i}{n}}}{2+3\left(2+\frac{10i}{n}\right)} = \frac{10\sqrt[3]{2+\frac{10i}{n}}}{8n+30i},$$

and so we have the area with n rectangles as $R_n = \sum_{i=1}^n \frac{10\sqrt[3]{2+\frac{10i}{n}}}{8n+30i}$.

Since the exact value of the integral is the limit as the number of rectangles goes to infinity, we have

$$\int_2^{12} \frac{\sqrt[3]{x}dx}{2+3x} = \lim_{n\to\infty} \sum_{i=1}^n \frac{10\sqrt[3]{2+\frac{10i}{n}}}{8n+30i}$$

4. Let $f(w) = \dfrac{w^4 - 2w^2\sqrt{w}}{w}$.

(a) Find $f'(w)$.

Solution: Observe that $f(w) = w^3 - 2w^{3/2}$. So $f'(w) = 3w^2 - 3w^{1/2}$.

(b) Find $\int f(w)dw$.

Solution:

$$\int f(w)dw = \int w^3 - 2w^{3/2}dw = \frac{1}{4}w^4 - 2\cdot\frac{2}{5}w^{5/2} + C = \frac{w^4}{4} - \frac{4w^{5/2}}{5} + C$$

5. Simplify.

(a) $\frac{d}{dx}\left(\int_{\pi}^{e^x} 3\cos t dt\right)$.

Solution: We use the fundamental theorem of calculus, and remember that we must use the chain rule (since the upper bound of integration is not x):

$$\frac{d}{dx}\left(\int_{\pi}^{e^x} 3\cos t dt\right) = 3\cos(e^x)\cdot(\frac{d}{dx}e^x) = 3e^x\cos(e^x)$$

(b) $\int (2x^{-1} + 7\sin x)\ dx$

Solution:

$$\int (2x^{-1} + 7\sin x)\ dx = 2\ln|x| - 7\cos(x) + C$$

6. The velocity of a particle at time t is given by $v(t) = 3t^2 - 1$. The position of the particle at time $t = 0$ is 1. Find the position of the particle at time $t = 3$.

Solution: We remember that if $s(t)$ is the position at time t, the velocity is $v(t) = s'(t)$. We have to find an the antiderivative $s(t) = \int v(t)\ dt$ for which $s(0) = 1$.

$$s(t) = \int 3t^2 - 1\ dx = t^3 - t + C$$

Since $s(0) = 1$, we must have $C = 1$. Thus

$$s(t) = t^3 - t + 1.$$

This means the position at $t = 3$ is $s(3) = 27 - 3 + 1 = 25$.

7. Consider the following integral

$$\int_1^2 3 + 2x \; dx$$

(a) Express the integral as the limit of Riemann sums. Do not evaluate the limit. Do not use Δx or x_i^* in your final answer; instead, plug in the formulas for these.

Solution: First, we write the right approximation with n rectangles (right, left, mid-point, . . . any will do). Since we have $\Delta x = (2-1)/n = 1/n$, we also have $x_i^* = 1+i/n$. Thus,

$$R_n = \sum_{i=0}^{n} \left(\frac{1}{n}\right)\left(3 + 2\left(1 + \frac{i}{n}\right)\right) = \sum_{i=0}^{n} \left(\frac{5}{n} + \frac{2i}{n^2}\right)$$

Taking the limit as $n \to \infty$ gives us the integral:

$$\int_1^2 3 + 2x \; dx \quad = \quad \lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{5}{n} + \frac{2i}{n^2}\right)$$

(b) Use the formula $\displaystyle\sum_{i=1}^{n} i = \frac{n^2+n}{2}$ to evaluate the limit in the previous part. DO NOT CALCULATE THE INTEGRAL DIRECTLY, or you will get no credit.

Solution: In order to compute the limit, it will probably help to rearrange the above answer a bit first.

Observe that $\sum_1^n 1 = 1 + 1 + \ldots + 1 = n$. We have

$$\sum_{i=0}^{n} \left(\frac{5}{n} + \frac{2i}{n^2}\right) = \frac{5}{n}\sum_{i=0}^{n} 1 + \frac{2}{n^2}\sum_{i=0}^{n} i = \frac{5}{n} \cdot n + \frac{2}{n^2}\left(\frac{n^2+n}{2}\right) = 5 + \frac{n^2+n}{n^2}$$

Thus, we have

$$\int_1^2 3 + 2x \; dx = \lim_{n\to\infty} \left(5 + \frac{n^2+n}{n^2}\right) = 5 + 1 = 6.$$

Just to make sure we didn't do something stupid, we can check our answer by calculating the definite integral

$$\int_1^2 3 + 2x \; dx = 3x + x^2\Big|_1^2 = (6+4) - (3+1) = 6.$$

But doing that is worth no credit, just peace of mind.