MAT126 Fall 2009

Practice Final

The actual Final exam will consist of twelve problems

Problem 1 1. Evaluate $\int_{\pi/3}^{\pi/2} \sin^3(x)\cos^2(x)dx$

2. Evaluate $\int \sin^2(x)dx$

Solution:

1. First, recognize that the integrand consists of $\sin(x)$ raised to an odd power, multiplied by $\cos(x)$ raised to an even power. Such a function is a prime candidate for the substitution $u = \cos(x)$ (so that one of the $\sin(x)$ can be used for the change of variable, and the rest may be expressed in terms of $\cos(x)$ via the identity $\sin^2(x) = 1 - \cos^2(x)$).

Therefore we substitute $u = \cos(x)$, whereby $du = -\sin(x)dx$, and we can write (noting that $u = \cos(\pi/3) = 1/2$ when $x = \pi/3$, and $u = \cos(\pi/2) = 0$ when $x = \pi/2$)

$$\begin{aligned}
\int_{x=\pi/3}^{x=\pi/2} \sin^3(x)\cos^2(x)dx &= \int_{x=\pi/3}^{x=\pi/2} (-\sin^2(x)\cos^2(x))(-\sin(x)dx) \\
&= \int_{x=\pi/3}^{x=\pi/2} (-1+\cos^2(x))(\cos^2(x))(-\sin(x)dx) \\
&= \int_{u=1/2}^{u=0} (u^2-1)(u^2)du \\
&= \int_{u=1/2}^{u=0} (u^4-u^2)du \\
&= \frac{u^5}{5} - \frac{u^3}{3}\Big|_{u=1/2}^{u=0} \\
&= \left(\frac{(0)^5}{5} - \frac{(0)^3}{3}\right) - \left(\frac{(1/2)^5}{5} - \frac{(1/2)^3}{3}\right) \\
&= -\frac{2^{-5}}{5} + \frac{2^{-3}}{3}
\end{aligned}$$

2. There are two ways to approach this problem. The first is to use the identity

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$

to get

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

This implies that

$$\begin{aligned}\int \sin^2(x)dx &= \int \frac{1}{2}(1-\cos(2x))dx \\ &= \int \frac{1}{2}dx - \int \frac{1}{2}\cos(2x)dx \\ &= \frac{1}{2}x - \frac{1}{2}\int \cos(2x)dx \\ &= \frac{1}{2}x - \frac{1}{2}\frac{\sin(2x)}{2} + C \\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C\end{aligned}$$

The second approach is to use an integration by parts, setting $u = \sin x$ and $dv = \sin x dx$; this means that $du = \cos x dx$ and $v = -\cos x$. Therefore, integration by parts gives

$$\int \sin^2(x)dx = (\sin x)(-\cos x) - \int(-\cos x)(\cos x dx) = -\sin x \cos x + \int \cos^2(x)dx$$

Now we plug in the identity $\cos^2(x) = 1 - \sin^2(x)$ to get

$$\int \sin^2(x)dx = -\sin x \cos x + \int(1-\sin^2(x))dx = -\sin x \cos x + \int dx - \int \sin^2(x)dx$$

Adding $\int \sin^2(x)dx$ to both sides of the equation gives

$$2\int \sin^2(x)dx = -\sin(x)\cos(x) + \int dx = -\sin(x)\cos(x) + x + C$$

so that

$$\int \sin^2(x)dx = \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + C$$

Notice that both approaches give the same answer, because of the identity $\sin(2x) = 2\sin(x)\cos(x)$.

Problem 2 1. Estimate the integral

$$\int_7^8 \frac{dx}{\ln(x)}$$

using three rectangles and

(a) right endpoints

(b) left endpoints

(c) Are your answers in 1a and 1b over- or under-estimates of the actual integral?

2. Do the same for the under-integral function $f(t) = e^{t^3}$

Solution:

1. The problem asks for three rectangles, so we divide the interval $[7, 8]$ into three equal subintervals:

$$\left[7, 7\frac{1}{3}\right] \qquad \left[7\frac{1}{3}, 7\frac{2}{3}\right] \qquad \left[7\frac{2}{3}, 8\right]$$

Each subinterval has length $\frac{1}{3}$.

(a) For right endpoints, evaluate $f(x) = \frac{1}{\ln(x)}$ at the right endpoint of each subinterval, and multiply the sum by $\frac{1}{3}$, the length of the subintervals:

$$\frac{1}{3}\left[f\left(7\frac{1}{3}\right) + f\left(7\frac{2}{3}\right) + f(8)\right] = \frac{1}{3}\left(\frac{1}{\ln\left(7\frac{1}{3}\right)} + \frac{1}{\ln\left(7\frac{2}{3}\right)} + \frac{1}{\ln(8)}\right)$$

(b) For left endpoints, evaluate $f(x) = \frac{1}{\ln(x)}$ at the left endpoint of each subinterval, and multiply the sum by $\frac{1}{3}$, the length of the subintervals:

$$\frac{1}{3}\left[f(7) + f\left(7\frac{1}{3}\right) + f\left(7\frac{2}{3}\right)\right] = \frac{1}{3}\left(\frac{1}{\ln(7)} + \frac{1}{\ln\left(7\frac{1}{3}\right)} + \frac{1}{\ln\left(7\frac{2}{3}\right)}\right)$$

(c) First, notice that the function $\ln(x)$ is an increasing function of x, so that $f(x) = \frac{1}{\ln(x)}$ is a *decreasing* function of x. Since $f(x)$ is decreasing, the right endpoint $f(x_i)$ is always less than the other values of $f(x)$ in the subinterval $[x_{i-1}, x_i]$, and the left endpoint $f(x_{i-1})$ is always greater than the other values in the subinterval. Thus the right endpoint approximation from part (a) is an under-estimate, and the left endpoint approximation from part (b) is an over-estimate.

2. As in (1), we divide the interval $[7, 8]$ into three equal subintervals:

$$\left[7, 7\frac{1}{3}\right] \qquad \left[7\frac{1}{3}, 7\frac{2}{3}\right] \qquad \left[7\frac{2}{3}, 8\right]$$

of length $\frac{1}{3}$.

(a) For right endpoints, evaluate $f(x) = e^{t^3}$ at the right endpoint of each subinterval, and multiply the sum by $\frac{1}{3}$, the length of the subintervals:

$$\frac{1}{3}\left[f\left(7\frac{1}{3}\right) + f\left(7\frac{2}{3}\right) + f(8)\right] = \frac{1}{3}\left(e^{\left(7\frac{1}{3}\right)^3} + e^{\left(7\frac{2}{3}\right)^3} + e^{8^3}\right)$$

(b) For left endpoints, evaluate $f(x) = \frac{1}{\ln(x)}$ at the left endpoint of each subinterval, and multiply the sum by $\frac{1}{3}$, the length of the subintervals:

$$\frac{1}{3}\left[f(7) + f\left(7\frac{1}{3}\right) + f\left(7\frac{2}{3}\right)\right] = \frac{1}{3}\left(e^{7^3} + e^{\left(7\frac{1}{3}\right)^3} + e^{\left(7\frac{2}{3}\right)^3}\right)$$

(c) First, notice that the function x^3 is an increasing function of x, and e^x is also an increasing function of x, so that $f(x) = e^{x^3}$ is an increasing function of x. Since $f(x)$ is increasing, the right endpoint $f(x_i)$ is always greater than the other values of $f(x)$ in the subinterval $[x_{i-1}, x_i]$, and the left endpoint $f(x_{i-1})$ is always less than the other values in the subinterval. Thus the right endpoint approximation from part (a) is an over-estimate, and the left endpoint approximation from part (b) is an under-estimate.

Problem 3 Integrate

1.
$$\int \cos(\ln(x))dx$$

2.
$$\int e^{\sqrt{x}}dx$$

Solution:

1. The first thing to recognize is the presence of a composite function, $\cos(\ln(x))$, in the integrand. This immediately suggests a substitution $u = \ln(x)$, or $x = e^u$. This implies that $dx = e^u du$, and so we have

$$\int \cos(\ln(x))dx = \int \cos(u)dx = \int \cos(u) \cdot e^u du$$

Observe that the integrand is the product of two functions, e^u and $\cos(u)$, whose antiderivatives are not too exotic— therefore this integral is a candidate for integration by parts. Setting $w = \cos(u)$ and $dv = e^u du$, we get that $dw = -\sin(u)du$ and $v = e^u$, so integration by parts gives

$$\int \cos(u)e^u du = \cos(u)e^u - \int e^u(-\sin(u)du) = \cos(u)e^u + \int e^u \sin(u)du \quad (1)$$

The integral on the right is again a candidate for integration by parts, this time setting $w = \sin(u)$ and $dv = e^u du$, whereby $dw = \cos(u)du$ and $v = e^u$, so that

$$\int e^u \sin(u)du = e^u \sin(u) - \int e^u \cos(u)du \quad (2)$$

Plugging (2) into (1) gives

$$\int \cos(u)e^u du = \cos(u)e^u + \int e^u \sin(u)du = \cos(u)e^u + e^u \sin(u) - \int e^u \cos(u)du$$

Adding $\int e^u \cos(u)du$ to both sides gives

$$2\int \cos(u)e^u du = \cos(u)e^u + e^u \sin(u) + C$$

and we have

$$\int \cos(u)e^u du = \frac{1}{2}e^u(\cos(u) + \sin(u)) + C$$

Plugging back in our substitution $u = \ln(x)$, we have

$$\int \cos(\ln(x))dx = \frac{1}{2}e^{\ln(x)}(\cos(\ln(x)) + \sin(\ln(x))) + C = \frac{1}{2}x(\cos(\ln(x)) + \sin(\ln(x))) + C$$

2. As with part (1), notice first that the composite function $e^{\sqrt{x}}$ suggests that the substitution $u = \sqrt{x}$ may be useful. This substitution gives $x = u^2$, so that $dx = 2udu$. Therefore

$$\int e^{\sqrt{x}}dx = \int e^u dx = \int e^u \cdot 2udu$$

Since the derivative of $2u$ is simpler than $2u$ (a polynomial of lower degree; in this case, a constant), while the antiderivative of e^u is not more complicated than e^u (in this case, the anti-derivative of e^u is itself), this integral is a good candidate for integration by parts. Since we want to differentiate $2u$, take $w = 2u$ and $dv = e^u du$. This gives $dw = 2du$ and $v = e^u$, so integration by parts yields

$$\int e^u 2udu = e^u 2u - \int 2e^u du = 2ue^u - 2e^u + C = 2(u-1)e^u + C$$

Plugging back our substitution $u = \sqrt{x}$, this gives us

$$\int e^{\sqrt{x}}dx = 2(\sqrt{x} - 1)e^{\sqrt{x}} + C$$

Problem 4 Find the following indefinite integrals

1. $$\int \frac{dx}{(x+1)\sqrt{x}}$$

2. $$\int \frac{(e^x+1)dx}{e^x(e^x+2)}$$

Solution:

1. Here the substitution is more subtle; however, one can recognize that $\sqrt{x}$ and x are both easily expressed in terms of the variable $u = \sqrt{x}$, and that the presence of $\sqrt{x}$ in the denominator makes this a good candidate for substitution. Since

$$du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2}\frac{1}{\sqrt{x}}dx$$

we can write

$$\begin{aligned}\int \frac{dx}{(x+1)\sqrt{x}} &= \int \frac{2}{x+1}\frac{1}{2}x^{-1/2}dx \\ &= \int \frac{2}{u^2+1}du\end{aligned}$$

Now, notice that the resulting integral is one that we recognize: the anti-derivative of $\frac{1}{u^2+1}$ is $\arctan(u)$, so we get

$$\begin{aligned}\int \frac{dx}{(x+1)\sqrt{x}} &= \int \frac{2}{u^2+1}du \\ &= 2\arctan(u) + C \\ &= 2\arctan(\sqrt{x}) + C\end{aligned}$$

2. The simplest solution to this problem is to write

$$\int \frac{e^x+1}{e^x(e^x+2)}dx = \int \frac{e^x+1}{e^x+2}e^{-x}dx$$

and then substitute $u = e^{-x}$, with $du = -e^{-x}dx$. This gives

$$\begin{aligned}\int \frac{e^x+1}{e^x+2}(e^{-x}dx) &= \int \frac{u^{-1}+1}{u^{-1}+2}(-du) \\ &= -\int \frac{1+u}{1+2u}du\end{aligned}$$

by multiplying the top and bottom of the integrand by u (since $u = e^{-x} > 0$ for all x, this is legal). Using long division we find that

$$1 + u = \frac{1}{2}(1 + 2u) + \frac{1}{2}$$

so that

$$\begin{aligned}
\int \frac{e^x + 1}{e^x + 2}(e^{-x}dx) &= -\int \frac{1 + u}{1 + 2u}du \\
&= -\int \left(\frac{\frac{1}{2}(1 + 2u)}{1 + 2u} + \frac{\frac{1}{2}}{1 + 2u} \right) du \\
&= -\int \frac{1}{2}du - \int \frac{1}{2 + 4u}du \\
&= -\frac{1}{2}u - \frac{1}{4}\ln|2 + 4u| + C
\end{aligned}$$

using the substitution $w = 2+4u$, with $du = \frac{1}{4}dw$, in the rightmost integral (the absolute value sign is actually irrelevant, since $2 + 4u = 2 + 4e^{-x} > 0$ is positive for all x). We then substitute back $u = e^{-x}$ and get

$$\int \frac{e^x + 1}{e^x + 2}(e^{-x}dx) = -\frac{1}{2}e^{-x} - \frac{1}{4}\ln(2 + 4e^{-x}) + C$$

Note: It is perhaps more natural, at first glance, to try to substitute $u = e^x$ instead of e^{-x}. This means that $x = \ln(u)$, so that $dx = \frac{1}{u}du$, and this substitution gives

$$\begin{aligned}
\int \frac{e^x + 1}{e^x(e^x + 2)}dx &= \int \frac{u + 1}{u(u + 2)}dx \\
&= \int \frac{u + 1}{u(u + 2)}\frac{1}{u}du \\
&= \int \frac{u + 1}{u^2(u + 2)}du
\end{aligned}$$

This integral could be computed using partial fractions, but since the denominator has a double root (one of the factors is u^2), we won't go down this path.

Problem 5 Find the following indefinite integrals

1. $$\int \frac{x^2+1}{x^2-3x+2}dx$$

2. $$\int \frac{x^3+1}{x^2-4x+3}dx$$

Solution: Both of these integrals require the method of partial fractions. Remember that this technique requires that the numerator have degree *lower* than the degree of the denominator; this is not the case for either integral, so we must first use long division to decompose the numerator appropriately.

1. The first step, as mentioned above, is to use long division to expand the numerator. Since the highest (x^2) term has the same coefficient in the numerator and denominator, we know that

$$x^2+1 = 1 \cdot (x^2-3x+2) + (\text{degree 1 polynomial})$$

Comparing terms shows that $x^2+1-(x^2-3x+2) = 3x-1$, so that

$$x^2+1 = (x^2-3x+2)+(3x-1)$$

(it's a good idea to check this to verify the arithmetic!)

Therefore we can write

$$\frac{x^2+1}{x^2-3x+2} = \frac{(x^2-3x+2)+(3x-1)}{x^2-3x+2} = 1+\frac{3x-1}{x^2-3x+2}$$

which tells us that

$$\int \frac{x^2+1}{x^2-3x+2}dx = \int dx + \int \frac{3x-1}{x^2-3x+2}dx = x + \int \frac{3x-1}{x^2-3x+2}dx \qquad (3)$$

Now the problem is reduced to solving the integral on the right, in which the numerator has degree 1— lower than the denominator's 2— so partial fractions can be applied. Write

$$\frac{3x-1}{x^2-3x+2} = \frac{3x-1}{(x-2)(x-1)} = \frac{A}{x-2}+\frac{B}{x-1}$$

Putting the right hand side over common denominator $x^2 - 3x + 2$ and comparing the two numerators, we see that

$$3x - 1 = A(x - 1) + B(x - 2) = (A + B)x + (-A - 2B)$$

implying the system of equations

$$\begin{aligned} A + B &= 3 \\ -A - 2B &= -1 \end{aligned}$$

The solution of this system is $A = 5$, $B = -2$. Therefore we have our partial fractions decomposition

$$\frac{3x - 1}{x^2 - 3x + 2} = \frac{5}{x - 2} + \frac{-2}{x - 1}$$

and the integral

$$\int \frac{3x - 1}{x^2 - 3x + 2}dx = \int \left(\frac{5}{x - 2} + \frac{-2}{x - 1} \right) dx = 5 \ln |x - 2| - 2 \ln |x - 1| + C$$

Plugging this into (3) finally yields

$$\int \frac{x^2 + 1}{x^2 - 3x + 2}dx = x + (5 \ln |x - 2| - 2 \ln |x - 1|) + C$$

2. Proceed as in part (1) above, but note that the numerator now has degree 3, so there are two steps to the long division. First note that the leading term of the numerator (x^3) is x times the leading term of the denominator (x^2), so that

$$x^3 + 1 = x \cdot (x^2 - 4x + 3) + (\text{degree 2 polynomial})$$

and we subtract $x^3 + 1 - x(x^2 - 4x + 3)$ to find the degree 2 polynomial on the right

$$x^3 + 1 = x \cdot (x^2 - 4x + 3) + 4x^2 - 3x + 1$$

Now the remainder term on the right is still degree 2 (not degree lower than the denominator yet), so we apply long division to $4x^2 - 3x + 1$ to obtain

$$4x^2 - 3x + 1 = 4 \cdot (x^2 - 4x + 3) + 13x - 11$$

All together, we have the partial fraction decomposition

$$\frac{x^3 + 1}{x^2 - 4x + 3} = \frac{x(x^2 - 4x + 3) + 4(x^2 - 4x + 3) + 13x - 11}{x^2 - 4x + 3} = x + 4 + \frac{13x - 11}{x^2 - 4x + 3}$$

so that

$$\begin{aligned}\int \frac{x^3+1}{x^2-4x+3}dx &= \int\left(x+4+\frac{13x-11}{x^2-4x+3}\right)dx \\ &= \frac{x^2}{2}+4x+\int\frac{13x-11}{x^2-4x+3}dx \qquad (4)\end{aligned}$$

Next, we apply partial fractions to the integral on the right. Write

$$\frac{13x-11}{x^2-4x+3} = \frac{13x-11}{(x-3)(x-1)} = \frac{A}{x-3}+\frac{B}{x-1}$$

Again, putting the right side over a common denominator $x^2 - 4x + 3$ and comparing numerators, we get

$$13x - 11 = A(x-1) + B(x-3) = (A+B)x + (-A-3B)$$

and the system of equations

$$\begin{aligned} A + B &= 13 \\ -A - 3B &= -11 \end{aligned}$$

which has solution $A = 14$, $B = -1$. Therefore

$$\int\frac{13x-11}{x^2-4x+3}dx = \int\left(\frac{14}{x-3}+\frac{-1}{x-1}\right)dx = 14\ln|x-3| - \ln|x-1| + C$$

Plugging this into (4) gives the final answer

$$\int\frac{x^3+1}{x^2-4x+3}dx = \frac{x^2}{2}+4x+14\ln|x-3|-\ln|x-1|+C$$

Problem 6 1. Use trapezoidal approximation with $n = 4$ intervals of subdivision (each of equal length) to estimate the following integral

$$\int_0^{\pi} \sin^2(x)dx.$$

Estimate the error of approximation.

2. Use Simpson's rule with $n = 4$ to estimate $\int_0^1 x^5 dx$. Find the precise error of approximation.

Solution:

1. The trapezoidal approximation is given by

$$T_n = \frac{1}{2n}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))$$

In our case, $n = 4$, and therefore the interval $[0, \pi]$ is divided into 4 equal subintervals, with sample points $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$, $x_3 = 3\pi/4$, and $x_4 = \pi$. We evaluate $f(x) = \sin^2(x)$ at each of these sample points, and get

$$\begin{aligned}
T_4 &= \frac{1}{8}(f(0) + 2f(\pi/4) + 2f(\pi/2) + 2f(3\pi/4) + f(\pi)) \\
&= \frac{1}{8}(\sin^2(0) + 2\sin^2(\pi/4) + 2\sin^2(\pi/2) + 2\sin^2(3\pi/4) + \sin^2(\pi)) \\
&= \frac{1}{8}\left(0 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2(1)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 0\right) \\
&= \frac{1}{8}\left(2 \cdot \frac{1}{2} + 2 + 2 \cdot \frac{1}{2}\right) \\
&= \frac{1}{8}(4) = \frac{1}{2}
\end{aligned}$$

To estimate the error in this approximation, we need to estimate the second derivative $f''(x)$ for all x in the interval $[0, \pi]$. First calculate (using the Chain Rule):

$$\begin{aligned}
f(x) &= \sin^2(x) \\
f'(x) &= 2\sin(x)\cos(x) = \sin(2x) \\
f''(x) &= \cos(2x) \cdot 2 = 2\cos(2x)
\end{aligned}$$

Since $|\cos(2x)| \leq 1$ for any x (in particular, any x in our interval $[0, \pi]$), we have

$$|f''(x)| = |2\cos(2x)| = 2|\cos(2x)| \leq 2$$

for all x in our interval $[0, \pi]$, we can take $K = 2$ in the trapezoid error approximation formula

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(\pi - 0)^3}{12 \cdot 4^2} = \frac{\pi^3}{96}$$

2. The Simpson's rule approximation is given by

$$S_n = \frac{1}{3n}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

In our case, $n = 4$, and therefore the interval $[0, 1]$ is divided into 4 equal subintervals, with sample points $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, and $x_4 = 1$. We evaluate $f(x) = x^5$ at each of these sample points, and get

$$\begin{aligned} S_4 &= \frac{1}{12}(f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1)) \\ &= \frac{1}{12}((0)^5 + 4(0.25)^5 + 2(0.5)^5 + 4(0.75)^5 + (1)^5) \\ &= \frac{1}{12}(0 + .00390625 + .0625 + .94921875 + 1) \\ &= \frac{1}{12}(2.015625) \approx 0.168 \end{aligned}$$

To estimate the error in this approximation, we need to estimate the fourth derivative $f^{(4)}(x)$ for all x in the interval $[0, 1]$. It is straightforward to see that the fourth derivative of $f(x) = x^5$ is $f^{(4)}(x) = 5 \cdot 4 \cdot 3 \cdot 2x = 120x$. On the interval $[0, 1]$ this function is *increasing and non-negative*, and so achieves its maximum absolute value at the right endpoint of interval— i.e. at $x = 1$— where it takes the value $f^{(4)}(1) = 120 \cdot 1 = 120$.

Therefore we have

$$|f^{(4)}(x)| \leq |f^{(4)}(1)| = 120$$

for all x in our interval $[0, 1]$, and we can take $K = 120$ in the Simpson's rule error approximation formula

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{120(1-0)^5}{180 \cdot 4^4} = \frac{120}{180 \cdot 256} \approx 0.003$$

The actual value of the integral is easy to compute:

$$\int_0^1 x^5 dx = \left.\frac{x^6}{6}\right|_{x=0}^{x=1} = \frac{1}{6} - 0 = \frac{1}{6} \approx 0.167$$

Note that this differs by only 0.001 from our approximation; this error is within our estimate of 0.003.

Problem 7 1. Does the following integral converge? If yes, evaluate it:

$$\int_0^{+\infty} e^{-t}\cos^2(t)dt$$

2. Does the following integral converge? If yes, evaluate it:

$$\int_1^{+\infty} \frac{dt}{t(t+1)}$$

3. Does the following integral converge? If yes, evaluate it:

$$\int_1^{\infty} \frac{dx}{x[\ln x]^2}$$

Solution:

1. The integrand is everywhere continuous, so the problem is Type I, and we only have to worry about the convergence at the upper limit.

Notice that the integrand is non-negative, and since $|\cos(t)| \leq 1$, we have

$$0 \leq e^{-t}\cos^2(t) \leq e^{-t}$$

for all t. Therefore by comparison, since $\int_0^{+\infty} e^{-t}dt$ converges, we know that our integral $\int_0^{+\infty} e^{-t}\cos^2(t)dt$ also converges.

In order to compute what this integral converges to, it is best to first use the identity

$$\cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

so that we have

$$\int_0^{\infty} e^{-t}\cos^2(t) = \int_0^{\infty} \frac{1}{2}e^{-t}dt + \int_0^{\infty} \frac{1}{2}\cos(2t)e^{-t}dt$$

The first integral on the right hand side is relatively easy to evaluate:

$$\begin{aligned}
\frac{1}{2}\int_0^{\infty} e^{-t}dt &= \frac{1}{2}\lim_{R\to\infty}\int_0^{R} e^{-t}dt \\
&= \frac{1}{2}\lim_{R\to\infty}\left(-e^{-t}\right)\Big|_{t=0}^{t=R} \\
&= \frac{1}{2}\lim_{R\to\infty}(-e^{-R} - (-e^{-0})) \\
&= \frac{1}{2}\left(-\lim_{R\to\infty} e^{-R} + e^{0}\right) \\
&= \frac{1}{2}(-0+1) = \frac{1}{2}
\end{aligned}$$

The second integral requires an integration by parts; setting $u = \cos(2t)$ and $dv = e^{-t}dt$, we get $du = -2\sin(2t)dt$ and $v = -e^{-t}$, so integration by parts gives

$$\int e^{-t}\cos(2t)dt = -e^{-t}\cos(2t) - \int(-e^{-t})(-2\sin(2t)dt) = -e^{-t}\cos(2t) - 2\int e^{-t}\sin(2t)dt$$

Apply another integration by parts to the integral on the right with $u = \sin(2t)$ and $dv = e^{-t}dt$ (so that $du = 2\cos(2t)dt$ and $v = -e^{-t}$) to get

$$\begin{aligned}\int e^{-t}\cos(2t)dt &= -e^{-t}\cos(2t) - 2\left[(-e^{-t})(\sin(2t)) - \int(-e^{-t})(2\cos(2t)dt)\right] \\ &= -e^{-t}\cos(2t) + 2e^{-t}\sin(2t) - 4\int e^{-t}\cos(2t)dt\end{aligned}$$

which, after some algebraic rearranging, gives

$$\int e^{-t}\cos(2t)dt = \frac{1}{5}\left(-e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\right) + C$$

Therefore,

$$\begin{aligned}\int_0^\infty \frac{1}{2}\cos(2t)e^{-t}dt &= \frac{1}{2}\lim_{R\to\infty}\int_0^R \cos(2t)e^{-t}dt \\ &= \frac{1}{2}\lim_{R\to\infty}\frac{1}{5}\left(-e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\right)\Big|_{t=0}^{t=R} \\ &= \frac{1}{10}\left(\lim_{R\to\infty}(-e^{-R}\cos(2R) + 2e^{-R}\sin(2R)) - (-e^0\cos(0) + 2e^0\sin(0))\right) \\ &= \frac{1}{10}[(0) - (-1\cdot 1 + 2\cdot 1\cdot 0)] = \frac{1}{10}[1] = \frac{1}{10}\end{aligned}$$

Putting together the two pieces of the original integral yields

$$\int_0^\infty e^{-t}\cos^2(t) = \frac{1}{2} + \frac{1}{10} = 0.5 + 0.1 = 0.6$$

2. Once again the integrand is continuous on the interval of integration (it has discontinuities at $t = 0$ and $t = -1$, but neither of these is in our interval of integration $[0, \infty)$) and non-negative, and we notice furthermore that

$$\frac{1}{t(t+1)} = \frac{1}{t^2 + t} < \frac{1}{t^2}$$

for all $1 \le t < \infty$. Therefore, since $\int_1^\infty \frac{1}{t^2}dt$ converges, we know by comparison that our integral $\int_1^\infty \frac{1}{t(t+1)}dt$ converges.

In order to calculate this integral, we need to use a partial fractions decomposition. The numerator 1 is already of lower degree than the denominator $(t^2 + t)$, which— conveniently— is already factored. Therefore set

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

and put the right hand side over the common denominator $t(t + 1)$, and then compare numerators of left and right hand sides to get

$$1 = A(t+1) + Bt = (A+B)t + A$$

which yields the system of equations

$$\begin{aligned} A + B &= 0 \\ A &= 1 \end{aligned}$$

with solution $A = 1$, $B = -1$. Therefore

$$\int \frac{1}{t(t+1)} dt = \int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = \ln|t| - \ln|t+1| + C = \ln\left|\frac{t}{t+1}\right|$$

Returning to our problem, calculate

$$\begin{aligned} \int_1^{+\infty} \frac{dt}{t(t+1)} &= \lim_{R\to\infty} \int_1^R \frac{dt}{t(t+1)} \\ &= \lim_{R\to\infty} \left(\ln\left|\frac{t}{t+1}\right|\right)\Big|_{t=1}^{t=R} \\ &= \lim_{R\to\infty} \left(\ln\frac{R}{R+1} - \ln\frac{1}{1+1}\right) \\ &= \ln\left(\lim_{R\to\infty} \frac{R}{R+1}\right) - \ln\left(\frac{1}{2}\right) \\ &= \ln(1) - (-\ln(2)) = 0 + \ln(2) = \ln(2) \end{aligned}$$

3. It is not easy to test the convergence of this integral without calculating the limit directly, so let's attack the integral itself. The integrand is continuous at all points in the interval $[1, \infty)$ except for $x = 1$, where $\ln^2(1) = 0$ and the denominator goes to zero. Therefore, *if* (and only if) the integral converges, it is defined by the limit

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x\ln^2(x)} &= \int_1^2 \frac{dx}{x\ln^2(x)} + \int_2^{\infty} \frac{dx}{x\ln^2(x)} \\ &= \lim_{R\to 1} \int_R^2 \frac{dx}{x\ln^2(x)} + \lim_{R\to\infty} \int_2^{\infty} \frac{dx}{x\ln^2(x)} \end{aligned}$$

In either case, we can integrate $\frac{1}{x\ln^2(x)}$ by the substitution $u = \ln(x)$, with $du = \frac{1}{x}dx$, giving

$$\begin{aligned}\int \frac{dx}{x\ln^2(x)} &= \int \frac{1}{\ln^2(x)}\frac{1}{x}dx \\ &= \int u^{-2}du \\ &= -u^{-1} + C \\ &= -(\ln(x))^{-1} + C\end{aligned}$$

Let's first check if the integral converges near $x = 1$. Recall we have

$$\begin{aligned}\int_1^2 \frac{dx}{x\ln^2(x)} &= \lim_{R\to 1}\int_R^2 \frac{dx}{x\ln^2(x)} \\ &= \lim_{R\to 1}[(-\ln(2))^{-1} - (-\ln(R))^{-1})] \\ &= (-\ln(2))^{-1} + \lim_{R\to 1}(\ln(R))^{-1}\end{aligned}$$

Now as $R \to 1$, we have $\ln(R) \to 0$, and so $(\ln(R))^{-1} \to \infty$ does not converge as $R \to 1$. Therefore, the integral $\int_1^2 \frac{dx}{x\ln^2(x)}$ does *not* converge. This means that the full integral $\int_1^\infty \frac{dx}{x\ln^2(x)}$ does *not* converge— regardless of whether the second part $\int_2^\infty \frac{dx}{x\ln^2(x)}$ converges (which it does)!

So, the answer is NO, the integral does not converge.

Problem 8 1. Find the volumes of the bodies obtained from the region enclosed by

$$y = \sin(x), 0 \le x \le \pi$$

and $y = 0$ by revolving about a) the x-axis, b) the line $x = -2$

2. Find the volumes of the bodies obtained from the region enclosed by

$$y = \frac{1}{x^5}, 1 \le x < \infty,$$

$y = 0$, $x = 1$ by revolving about a) the line $y = -2$, b) the y-axis.

Solution:

1. First step: DRAW A PICTURE!! (see accompanying diagrams)

Since $y = \sin(x)$ is not a pleasant function to invert, it's a good idea to integrate over x using vertical strips (notice, for example, that horizontal strips go from one arcsin(y) to another arcsin(y)... not something that we want to start having to compute).

(a) Each value of x between 0 and π gives a vertical strip, of thickness dx, that generates a disk when rotated about the x-axis. Thus we use the formula for volume of rotation via the disk method,

$$V = \int_0^\pi \pi R^2 dx$$

In our case, the radius of the disk is given by the distance from $(x, \sin x)$ to $(x, 0)$ for each x, and is therefore given by $R = \sin x$. Hence

$$V = \int_0^\pi \pi(\sin x)^2 dx$$

We can compute this integral by applying the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ (see Problem 1, part (2)):

$$\begin{aligned} V &= \pi \int_0^\pi \sin^2(x) dx = \pi \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right) \Bigg|_{x=0}^{x=\pi} \\ &= \pi \left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4} - \frac{0}{2} + \frac{\sin(0)}{4} \right) \\ &= \pi \left(\frac{\pi}{2} - 0 - 0 + 0 \right) = \frac{\pi^2}{2} \end{aligned}$$

(b) Since we are now rotating about the (vertical) line $x = -2$, our vertical strips no longer generate disks, but instead generate cylindrical shells. The formula for the volume of rotation via the cylindrical shell method is

$$V = \int_0^{\pi} 2\pi \cdot R \cdot h dx$$

Here R is the radius of rotation, given by the distance from x to the line of rotation at -2; this distance is $R = x + 2$. The height h is the height of the vertical strip, equal to $\sin x$ as before in part (a). Thus

$$\begin{aligned} V &= \int_0^{\pi} 2\pi(x+2)\sin(x)dx \\ &= 2\pi \int_0^{\pi} x\sin(x)dx + 2\pi \int_0^{\pi} 2\sin(x)dx \\ &= 2\pi \int_0^{\pi} x\sin(x)dx - 4\pi\cos(x)|_{x=0}^{x=\pi} \\ &= 2\pi \int_0^{\pi} x\sin(x)dx - (4\pi\cdot(-1) - 4\pi\cdot(1)) \\ &= 2\pi \int_0^{\pi} x\sin(x)dx + 8\pi \end{aligned}$$

The integral $\int_0^{\pi} x\sin(x)dx$ can be evaluated by an integration by parts, setting $u = x$ and $dv = \sin(x)dx$, whereby $du = dx$ and $v = -\cos(x)$. This gives

$$\begin{aligned} \int_0^{\pi} x\sin(x)dx &= -x\cos(x)|_{x=0}^{x=\pi} - \int_0^{\pi}(-\cos(x))dx \\ &= -\pi\cos(\pi) - 0 + \int_0^{\pi}\cos(x)dx \\ &= -\pi\cdot(-1) + \sin(x)|_{x=0}^{x=\pi} \\ &= \pi + (0-0) = \pi \end{aligned}$$

Putting all this together, we have

$$V = 2\pi \int_0^{\pi} x\sin(x)dx + 8\pi = 2\pi(\pi) + 8\pi = 2\pi^2 + 8\pi$$

2. As in part (1), we will use the disk/washer method for part (a) and the cylindrical shell method for part (b); although the function $y = x^{-5}$ is more easily inverted than the function in part (a), and we could exchange disk/washer and cylindrical shell methods by considering $x = y^{-1/5}$, it seems easier to integrate over x since we are given the information about the region in terms of y as a function of x.

(a) We have as above vertical strips stretching from (x, x^{-5}) to $(x, 0)$ for each x in $[0, \infty)$. However, unlike part (a), the axis of rotation is now $y = -2$, not the x-axis; therefore the region under consideration does NOT border the axis of rotation, and we have to use the washer method

$$V = \int_1^\infty (\pi R^2 - \pi r^2) dx$$

Here R is the outer radius of rotation, given by the distance of the graph of the function $y = x^{-5}$ from the line $y = -2$; so $R = x^{-5} + 2$. The inner radius r is the distance from the inner curve $y = 0$ to the line $y = -2$, which is $r = 2$ for all x. Therefore

$$\begin{aligned}
V &= \pi \int_1^\infty ((x^{-5} + 2)^2 - 2^2) dx \\
&= \pi \int_1^\infty (x^{-10} + 2x^{-5} + 4 - 4) dx \\
&= \pi \int_1^\infty (x^{-10} + 2x^{-5}) dx \\
&= \pi \left(\frac{x^{-9}}{-9} + 2\frac{x^{-4}}{-4} \right) \Bigg|_{x=1}^{x=\infty} \\
&= \pi(0 + 0 - (-1/9 - 1/4)) = \pi(1/9 + 1/4) = \frac{13\pi}{36}
\end{aligned}$$

(b) Once again we use the cylindrical shells formula

$$V = \int_1^\infty 2\pi \cdot R \cdot h dx$$

where R is the radius of rotation, given by the distance from x to the y-axis; therefore $R = x$. The height is given by the height of the vertical strip which, as in part (a), is given by $h = x^{-5}$.

Thus

$$\begin{aligned}
V &= 2\pi \int_1^\infty x \cdot x^{-5} dx = 2\pi \int_1^\infty x^{-4} dx \\
&= 2\pi \left(\frac{x^{-3}}{-3} \Bigg|_{x=1}^{x=\infty} \right) \\
&= 2\pi \left(0 - \frac{1}{-3} \right) = \frac{2\pi}{3}
\end{aligned}$$

Problem 9 Find the length of the curve given by the graph of the function

$$y = \ln(\cos(x))$$

between points $(0, 0)$ and $(a, \ln(\cos(a)))$, $0 < a < \frac{\pi}{2}$.

Solution: Observe that the starting point $(0, 0)$ of the curve corresponds to $x = 0$ (and $y = \ln(\cos(0)) = \ln(1) = 0$), and the terminal point $(a, \ln(\cos(a)))$ corresponds to $x = a$. Since the curve is parametrized by x, we have the arc length formula

$$L = \int_{x=0}^{x=a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

So the next step is to differentiate

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(\ln(\cos(x))) \\
&= \frac{1}{\cos(x)} \cdot \frac{d}{dx}(\cos(x)) \\
&= \frac{1}{\cos(x)} \cdot (-\sin(x)) \\
&= \frac{-\sin(x)}{\cos(x)} = -\tan(x)
\end{aligned}$$

using the Chain Rule.

Plugging this into the arc length formula, and recalling that

$$1 + \tan^2(x) = \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

we get that the arc length is

$$\begin{aligned}
L &= \int_0^a \sqrt{1 + (-\tan(x))^2} dx \\
&= \int_0^a \sqrt{1 + \tan^2(x)} dx \\
&= \int_0^a \sqrt{\sec^2(x)} dx \\
&= \int_0^a |\sec(x)| dx \\
&= \int_0^a \sec(x) dx
\end{aligned}$$

since $0 \leq x \leq a < \pi/2$ implies that $\sec(x) \geq 0$ on this interval.

Therefore

$$
\begin{aligned}
L &= \int_0^a \sec(x)dx \\
&= (\ln|\sec(x) + \tan(x)|)\Big|_{x=0}^{x=a} \\
&= \ln(\sec(a) + \tan(a)) - \ln(\sec(0) + \tan(0)) \\
&= \ln(\sec(a) + \tan(a)) - \ln(1 + 0) = \ln(\sec(a) + \tan(a)) - \ln(1) \\
&= \ln(\sec(a) + \tan(a))
\end{aligned}
$$

Problem 10 1. A cable that weights 2lb/ft is used to lift 800lb of coal from a mineshaft 500 ft deep. Find the work done.

2. A 10 ft chain weights 25 lb and hangs from the ceiling. Find the work done in lifting the middle of the chain to the ceiling so that it is level with the upper end.

Solution:

1. There are two parts to the work done: the work done in lifting the coal out of the mine, and the work done in lifting the chain itself out of the mine.

For the first part, the problem states that 800 lbs. of coal are lifted 500 ft., and so the total work done to lift the coal up out of the mine is

$$W_1 = (800 \text{ lbs. }) \cdot (500 \text{ ft. }) = 400,000 \text{ ft.-lbs.}$$

For the second part, we have to do an integral. Let x be the depth at which each piece of the chain is located— this means $0 \leq x \leq 500$. Each piece of chain of length dx has weight $(2 \text{ lb/ft. })(dx \text{ ft. }) = 2dx$ lbs., and is lifted a distance x to the top of the mine. Therefore the work done in lifting the chain out of the mine is given by the integral

$$W_2 = \int_0^{500} 2dx \cdot x = \int_0^{500} 2x dx = x^2|_{x=0}^{x=500} = 250,000 \text{ ft.-lbs.}$$

So the total work done is

$$W_1 + W_2 = 400,000 \text{ ft.-lbs. } + 250,000 \text{ ft.-lbs. } = 650,000 \text{ ft.-lbs.}$$

2. Here again, there are multiple parts to the work done. If the middle of the chain is lifted to be even with the top, then the top $1/4$ of the chain— the first 2.5 ft.— is not moved at all. On the other hand, the bottom half (5 ft.) of the chain is lifted a total distance of 5 ft.; or, more precisely, each piece of the bottom half of the chain is lifted to a point 5 ft. above its original position. Therefore, since the bottom half of the chain weighs $\frac{1}{2} \cdot 25$ lbs. $= 12.5$ lbs. , the work done in moving the bottom half of the chain 5 ft. upward is

$$W_1 = (12.5 \text{ lbs. }) \cdot (5 \text{ ft. }) = 62.5 \text{ ft.-lbs.}$$

It remains to examine the second $\frac{1}{4}$ of the chain. Here, each piece of the chain is lifted a distance equal to twice its original distance from the point P located $\frac{1}{4}$ of the way down the chain (eg., the point P does not move at all, but the midpoint of the chain starts 2.5 ft. below P, and ends up 2.5 ft. above P, which is $2 \cdot 2.5$ ft. $=$ 5 ft. above its original position). Therefore, if we set x to be the starting distance below P, then we let x range over $0 \leq x \leq 2.5$ ft., and each piece of thickness dx is moved upward a distance $2x$. The weight density of the chain is given by

$$\frac{25 \text{ lbs.}}{10 \text{ ft.}} = 2.5 \text{ lbs./ft.}$$

This means that a piece of chain of length dx will have weight $2.5dx$ lbs., and since it is moved a distance $2x$, the work done is given by the integral

$$\begin{aligned} W_2 &= \int_0^{2.5} (2.5dx)(2x) = \int_0^{2.5} 5x dx \\ &= 2.5x^2 \Big|_{x=0}^{x=2.5} = 2.5 \cdot (2.5)^2 - 0 \\ &= 15.625 \text{ ft.-lbs.} \end{aligned}$$

Putting the two parts of the work together, the total work done is

$$W_1 + W_2 = 62.5 \text{ ft.-lbs. } + 15.625 \text{ ft.-lbs. } = 78.125 \text{ ft.-lbs.}$$

Problem 11 A granary has the shape of a half cylinder lying on its rectangular side (the cut). The cylinder's height is 10m, and the radius of the base is 2m. If the granary is full of barley, with density $600 kg/m^3$, how much work is done in removing all the grain via an opening at the top of the granary?

Solution: Step 1— DRAW A PICTURE!! (see accompanying diagrams).

We divide the granary into horizontal slices, each at a height z above the base of the granary (so that z runs from 0 to 2m). For each z, the slice at height z is a rectangle of length 10m (the "height" of the sideways cylinder), and thickness dz, whose width stretches from one side of the semi-circular cross-section to the opposite side. A simple application of the Pythagorean Theorem shows that this width is $2\sqrt{4-z^2}$. Thus the volume of the slice at height z is given by

$$dV = (10\text{ m})(2\sqrt{4-z^2}\text{ m})(dz\text{ m}) = 20\sqrt{4-z^2}dz\text{ m}^3$$

Since the density of the barley is 600 kg/m^3, the mass of barley in this slice is

$$dm = (600\text{ kg/m}^3)(dV) = 600\cdot 20\sqrt{4-z^2}dz\text{ kg} = 12,000\sqrt{4-z^2}dz\text{ kg}$$

From the law $F = mg = m\cdot 9.8$, we have a force

$$dF = dm\cdot 9.8 = (9.8)(12,000\sqrt{4-z^2}dz) = 117,600\sqrt{4-z^2}dz\text{ N}$$

acting on the slice at height z.

Next, observe that the slice at height z is moved a distance $2-z$ upward to get out the top of the granary. Therefore,

$$W = \int_0^2 (117,600\sqrt{4-z^2}dz)\cdot(2-z) = 117,600\int_0^2 (2-z)\sqrt{4-z^2}dz$$

This integral can be split into two parts. The first part is

$$\begin{aligned} W_1 &= 117,600\int_0^2 2\sqrt{4-z^2}dz \\ &= 235,200\int_0^2 \sqrt{2^2-z^2}dz \end{aligned}$$

Observe that this integral is a candidate for the trigonometric substitution $z = 2\sin(u)$, with $dz = 2\cos(u)du$, so that

$$\int_{z=0}^{z=2} \sqrt{2^2 - z^2}dx = \int_{u=0}^{u=\pi/2} \sqrt{2^2 - 2^2\sin^2(u)}(2\cos(u)du) = \int_0^{\pi/2} 4\cos^2(u)du$$

since

$$\sqrt{2^2(1-\sin^2(u))} = 2\sqrt{\cos^2(u)} = \cos(u)$$

for $0 \le u \le \pi/2$, since $\cos(u) \ge 0$ on this interval.

Therefore, using the identity $\cos^2(u) = \frac{1}{2} + \frac{1}{2}\cos(2u)$, this work becomes

$$\begin{aligned}
W_1 &= 235,200 \int_{u=0}^{u=\pi/2} 4\cos^2(u)du \\
&= 235,200 \cdot 4 \left[\int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2u)\right) du\right] \\
&= 235,200 \cdot 4 \left(\frac{u}{2} + \frac{\sin(2u)}{4}\right)\Bigg|_0^{\pi/2} \\
&= 235,200 \cdot 4 \left(\frac{\pi/2}{2} + \frac{\sin(\pi)}{4} - \frac{0}{2} - \frac{\sin(0)}{4}\right) \\
&= 235,200 \cdot 4 \cdot \frac{\pi}{4} = 235,200\pi
\end{aligned}$$

The second part can be computed via the substitution $u = 4 - z^2$, with $du = -2zdz$:

$$\begin{aligned}
W_2 &= 117,600 \int_{z=0}^{z=2} (-z)\sqrt{4 - z^2}dz \\
&= 58,800 \int_{z=0}^{z=2} \sqrt{4 - z^2}(-2zdz) \\
&= 58,800 \int_{u=4}^{u=0} \sqrt{u}du \\
&= 58,800 \left.\frac{u^{3/2}}{3/2}\right|_{u=4}^{u=0} \\
&= 58,800 \left(0 - \frac{2}{3}4^{3/2}\right) \\
&= -58,800 \cdot \frac{2}{3} \cdot 8 = -313,600
\end{aligned}$$

Thus the total work done is

$$W_1 + W_2 = 235,200\pi - 313,600$$

Problem 12 If $f(x)$ is an increasing function on $[0,1]$, rank the following in order from least to greatest:

- f(0)
- f(1)
- The left endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- The right endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- The average value of f on $[0,1]$.

Solution: The first step is to reinterpret some of the quantities listed here. Notice that, since the interval of integration has length 1, the average value of f on $[0,1]$ is given by

$$f_{\text{avg}} = \frac{1}{1-0}\int_0^1 f(t)dt = \int_0^1 f(t)dt$$

We know that, whenever f is an increasing function, the left endpoint of an interval gives a value lower than the other values in the interval, and the right endpoint gives a value greater than the other values in the interval; thus the left endpoint approximation will be an underestimate for the integral, and the right endpoint will be an overestimate. Therefore, we have

$$\text{left endpoint approximation} < \text{average value of } f < \text{right endpoint approximation}$$

It remains to see where $f(0)$ and $f(1)$ fit in to the heirarchy. Notice that the left endpoint approximation is given by

$$L_5 = \frac{1}{5}(f(0) + f(1) + f(2) + f(3) + f(4))$$

which is the average of the values $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$. Since each of these is at least as big as $f(0)$, and the latter 4 are greater than $f(0)$ (since f is increasing), this left endpoint approximation is greater than $f(0)$. Similarly, the right endpoint approximation is the average of $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$— all of which are no greater than $f(1)$, and the first 4 of which are strictly less than $f(1)$. Thus the right endpoint approximation is less than $f(1)$.

Summarizing, we have the following order from least to greatest:

- f(0)
- The left endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- The average value of f on $[0, 1]$.
- The right endpoint approximation to $\int_0^1 f(x)dx$ with $n = 5$ rectangles.
- f(1)

It may be useful to interpret these quantities as given by the areas of rectangles over the x-axis (see attached figure). The value $f(0)$ is equal to the area of a rectangle streching from 0 to 1 of height $f(0)$, which lies entirely underneath all points on the graph of f from 0 to 1. The left endpoint approximation consists of rectangles each lying underneath the graph *in each respective subinterval*, though all but the leftmost stretch above $f(0)$. Similarly on the other extreme, the right endpoint approximation is given by the areas of rectangles, each stretching above the graph in each subinterval, though all but the rightmost lie underneath the value $f(1)$. The value $f(1)$ corresponds to a rectangle of length 1 and height $f(1)$, which is entirely above all points in the graph of f (and rectangles of the right endpoint approximation).

Problem 13 For each of the following, determine if the improper integral converges or diverges. If it converges, evaluate the integral.

1. $\int_{-\infty}^{\infty} 2xe^{-x^2} dx$
2. $\int_{-\infty}^{\infty} \frac{1}{2x} e^{-x^2} dx$

Solution:

1. First, observe that the integrand is everywhere continuous, so we do not have to worry about any "Type II" convergence issues; the only thing to check is whether or not the integral converges out to $\pm\infty$. For this, split the integral into two parts, one for each (infinite) limiting endpoint:

$$\begin{aligned}
\int_{-\infty}^{\infty} 2xe^{-x^2} dx &= \int_{-\infty}^{0} 2xe^{-x^2} dx + \int_{0}^{\infty} 2xe^{-x^2} dx \\
&= \lim_{R\to\infty} \int_{-R}^{0} 2xe^{-x^2} dx + \lim_{R\to\infty} \int_{0}^{R} 2xe^{-x^2} dx
\end{aligned}$$

if both of these limits exist.

Next, notice that we can compute these (finite) integrals with the substitution $u = -x^2$, since this gives $du = -2xdx$, and we have

$$\begin{aligned}
\int 2xe^{-x^2} dx &= -\int e^{-x^2} (-2xdx) \\
&= -\int e^u du = -e^u + C \\
&= -e^{-x^2} + C
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{0} 2xe^{-x^2} dx &= \lim_{R\to\infty} -e^{-x^2}\Big|_{x=-R}^{x=0} \\
&= \lim_{R\to\infty} \left(-e^{-0^2} + e^{-(-R)^2}\right) \\
&= -e^0 + \lim_{R\to\infty} -e^{-R^2} \\
&= -1 + 0 = -1
\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{R\to\infty}\int_0^R 2xe^{-x^2}dx &= \lim_{R\to\infty} -e^{-x^2}\Big|_{x=0}^{x=R} \\ &= \lim_{R\to\infty} -e^{-R^2}+e^{-0^2} \\ &= 0+1=1\end{aligned}$$

This means that both halves converge, and so our original integral from $-\infty$ to ∞ converges, and its value is equal to

$$\int_{-\infty}^{\infty} 2xe^{-x^2}dx = \lim_{R\to\infty}\int_{-R}^{0} 2xe^{-x^2}dx + \lim_{R\to\infty}\int_0^R 2xe^{-x^2}dx = -1+1=0$$

Thus the integral converges to 0.

2. Here there are two issues to worry about: not only is the interval of integration infinite, but the integrand is also discontinuous at 0, so we have both "Type I" and "Type II" convergence issues. Break up the integral into 4 pieces, one for each potential convergence problem:

$$\int_{-\infty}^{\infty}\frac{1}{2x}e^{-x^2}dx = \int_{-\infty}^{-1}\frac{1}{2x}e^{-x^2}dx + \int_{-1}^{0}\frac{1}{2x}e^{-x^2}dx + \int_{0}^{1}\frac{1}{2x}e^{-x^2}dx + \int_{1}^{\infty}\frac{1}{2x}e^{-x^2}dx$$

This integral is not easily evaluated; however, let's try to see if we can determine whether or not it converges. Near $\pm\infty$, the factor e^{-x^2} is very very small, and multiplying it by $\frac{1}{2x}$ only makes it smaller— this suggests that the first and last pieces should converge (a rigorous argument would use the Comparison Test, eg. comparing with the function e^x and e^{-x}, for x negative and x positive, respectively). On the other hand, near 0, the factor e^{-x^2} is close to a constant, while $\frac{1}{2x}$ gets very large, and the whole integral behaves like $\int \frac{\text{constant}}{x}dx$, which we know diverges.

So let's apply the Comparison Test to show that the integral from 0 to 1 diverges. For $0 \le x \le 1$, the function e^{-x^2} satisfies

$$e^{-x^2} \ge e^{-1} \ge \frac{1}{3}$$

Therefore

$$\int_0^1 \frac{1}{2x}e^{-x^2}dx \ge \int_0^1 \frac{1}{2x}\cdot\frac{1}{3}dx = \frac{1}{6}\int_0^1\frac{dx}{x}$$

But we know that this last integral diverges, since

$$\begin{aligned}\lim_{R\to 0}\int_R^1 \frac{dx}{x} &= \lim_{R\to 0} \ln(x)|_{x=R}^{x=1} \\ &= \ln(1) - \lim_{R\to 0}\ln(R) = -\lim_{R\to 0}\ln(R)\end{aligned}$$

and $\ln(R) \to -\infty$ (as $R \to 0$) does not converge.

Therefore, our original integral $\int_{-\infty}^{\infty} \frac{1}{2x} e^{-x^2} dx$ diverges.

Problem 14 A particle starts at the origin at time $t = 0$, and traces out a path given by

$$\begin{aligned} x(t) &= t \\ y(t) &= 2t^2 \end{aligned}$$

for each $t \geq 0$.

1. Express the length $l(T)$ of the path traced out by the particle from $t = 0$ to a time $t = T$ as an integral, but do not evaluate it.
2. Find $l'(T)$. (Side note: this is the speed of the particle at time T.)
3. Evaluate $l'(2)$.

Solution:

1. Since we are given the parametrization of the curve, this is simply given by the arc length formula

$$l(T) = \int_{t=0}^{t=T} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

So the next step is to evaluate the derivatives:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}(t) = 1 \\ \frac{dy}{dt} &= \frac{d}{dt}(2t^2) = 4t \end{aligned}$$

Therefore

$$l(T) = \int_{t=0}^{t=T} \sqrt{1 + (4t)^2} dt = l(T) = \int_{t=0}^{t=T} \sqrt{1 + 16t^2} dt$$

2. Since $l(T)$ is defined as the integral of $f(t) = \sqrt{1 + 16t^2}$ from 0 to T, the Fundamental Theorem of Calculus implies that

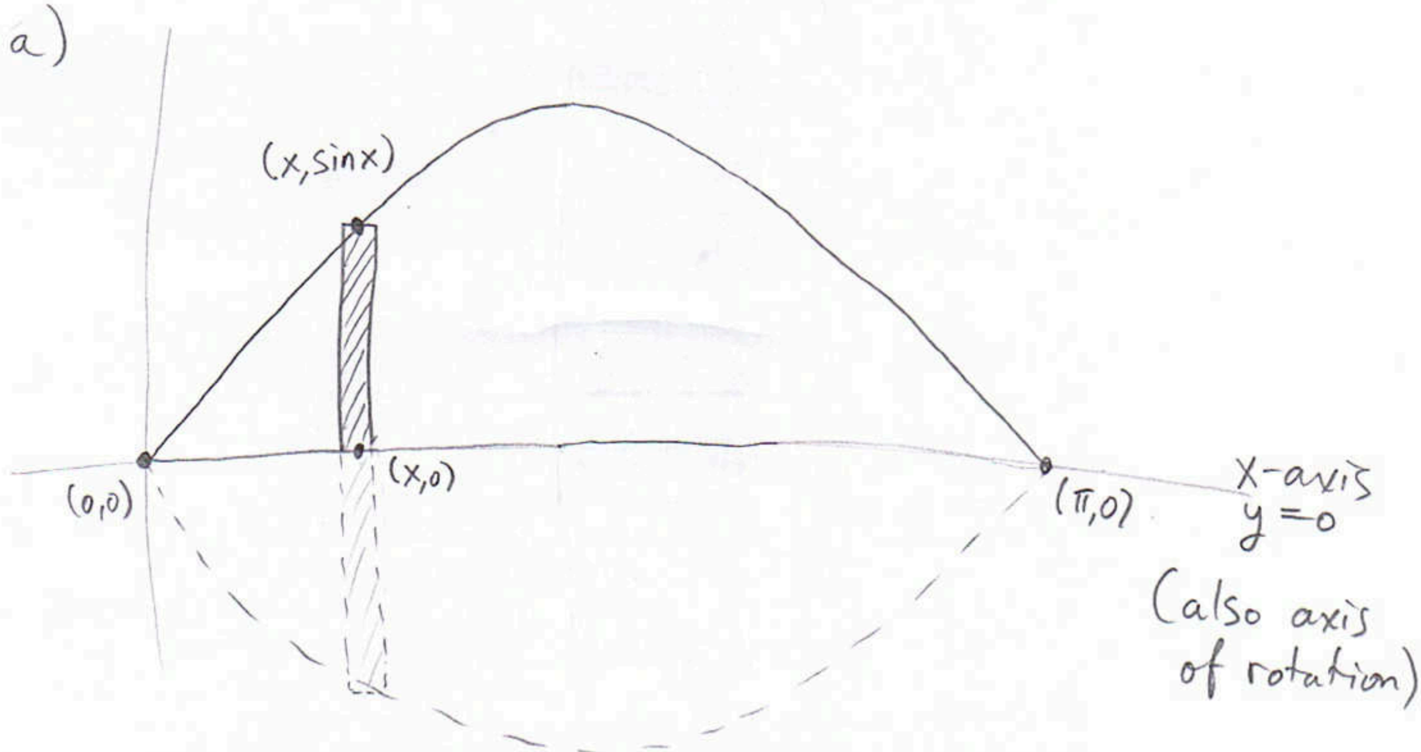
$$l'(T) = f(T) = \sqrt{1 + 16T^2}$$

3. Since part (2) gave us the function $l'(T) = \sqrt{1 + 16T^2}$, we simply evaluate this function at the point $T = 2$, giving

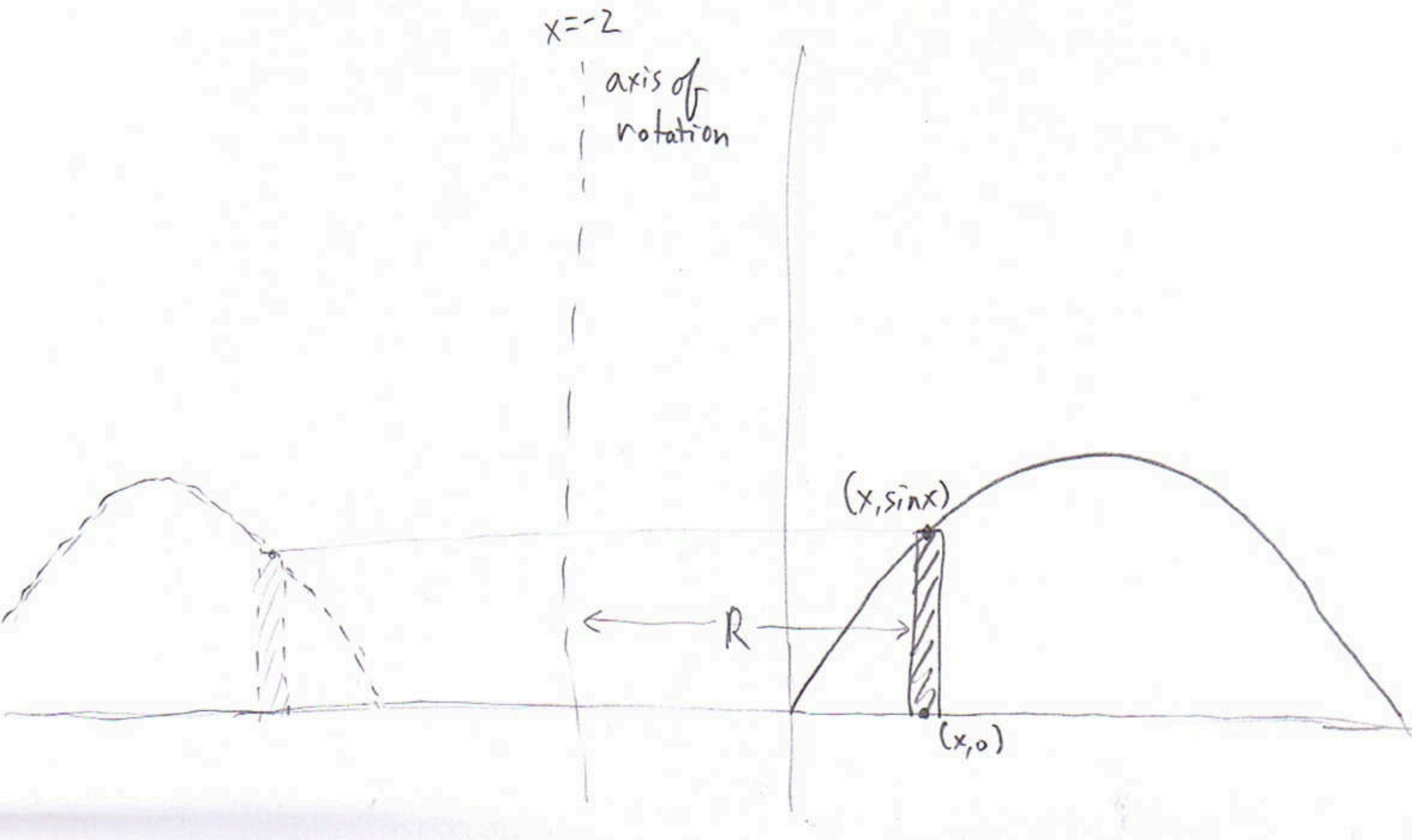
$$l'(2) = \sqrt{1 + 16(2)^2} = \sqrt{1 + 16 \cdot 4} = \sqrt{65}$$

Problem 8

1. a)

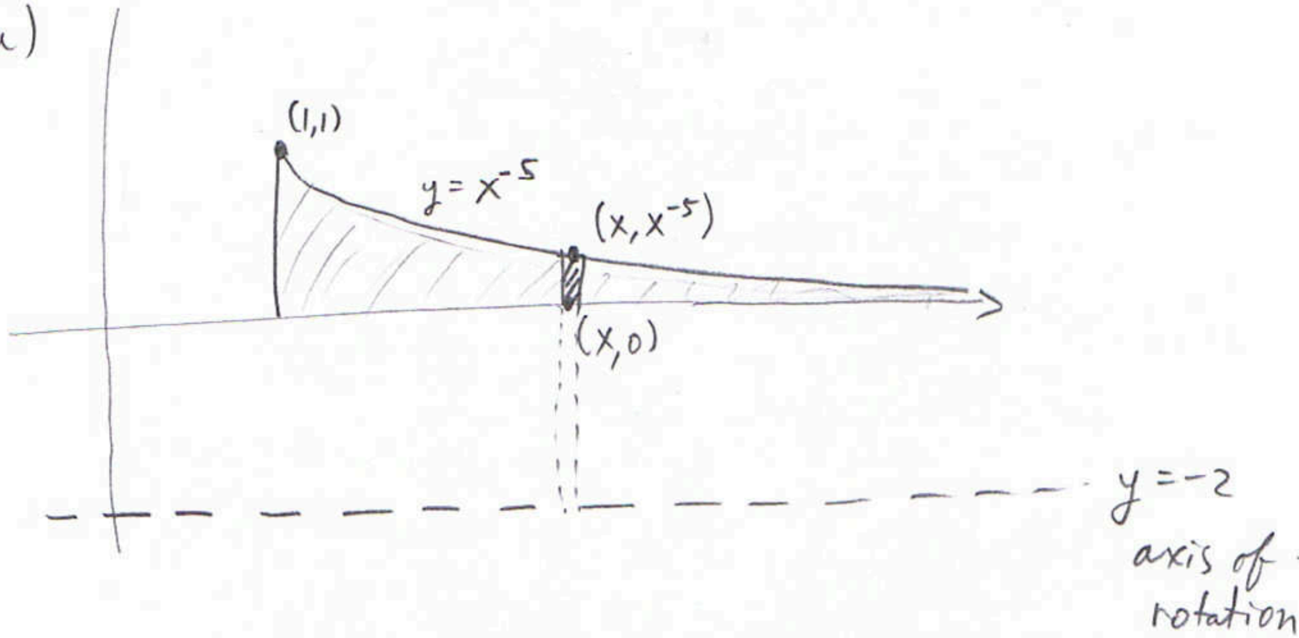


1. b)

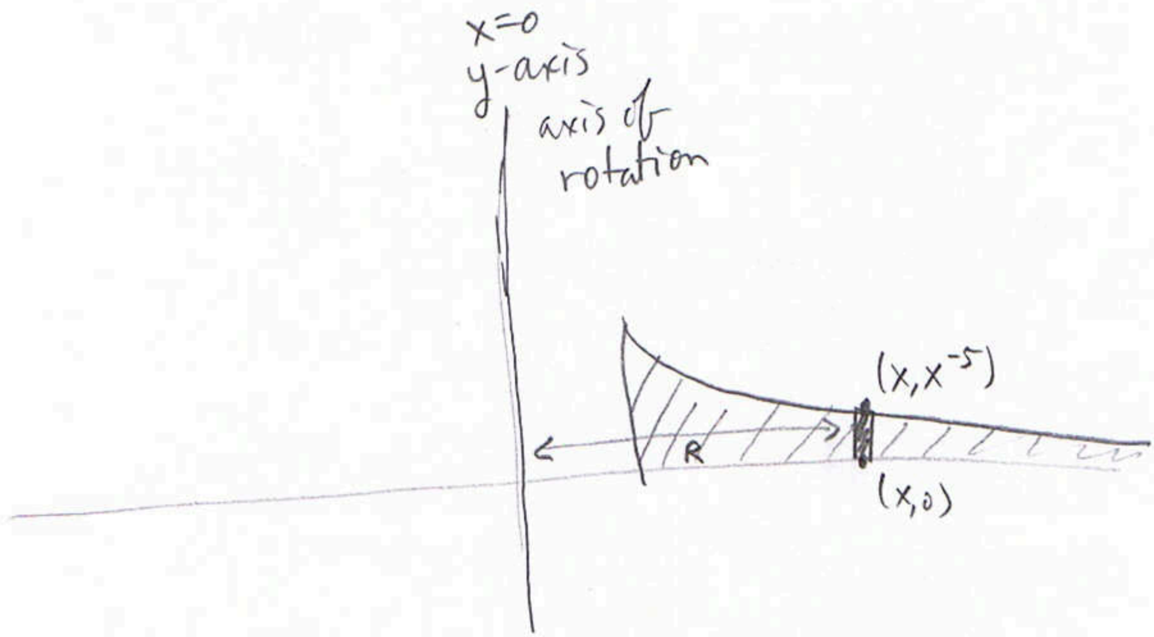


Problem 8

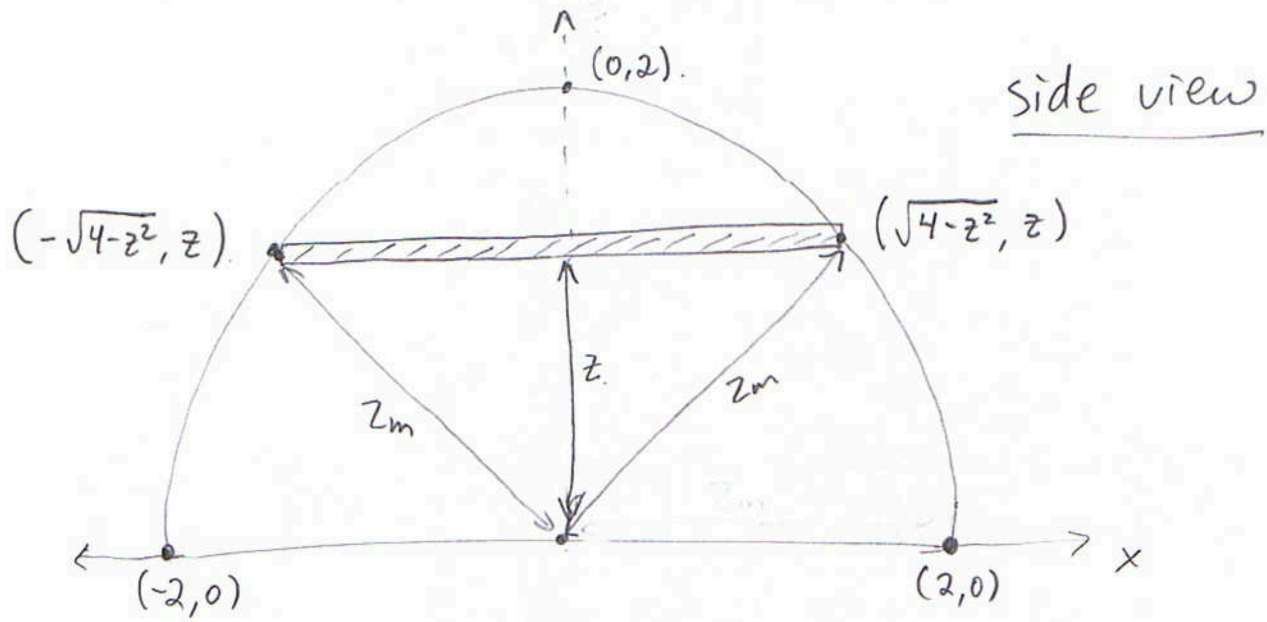
2. (a)



(b)



Problem 11



top view of each slice: