MAT 126 Solutions to Midterm 2 (Tweedledum)

1. Compute each of the following. If the integral does not converge, write "Diverges". You must justify your answer to receive full credit.

8 pts (a) $\displaystyle\int_1^{e^\pi} \frac{\sin(\ln x)\big(\cos(\ln x)\big)^6}{x}\,dx$

Solution: Make the substitution $u = \cos(\ln x)$ so that $du = \dfrac{-\sin(\ln x)}{x}\,dx$. Then we have

$$\int_1^{e^\pi} \frac{\sin(\ln x)\big(\cos(\ln x)\big)^6}{x}\,dx = -\int_1^{e^\pi} u^6\,du = \frac{-(\cos\ln x)^7}{7}\Bigg|_1^{e^\pi} = \frac{-(\cos\pi)^7 + (\cos 0)^7}{7} = \boxed{\frac{2}{7}}.$$

8 pts (b) $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$

Solution: Since one of the integration limits is infinite, this is an improper integral, so we must evaluate it as a limit. We have

$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{B\to\infty}\int_2^B \frac{1}{x^2}\,dx = \lim_{B\to\infty}\frac{-1}{x}\Bigg|_2^B = \lim_{B\to\infty}\frac{-1}{B} + \frac{1}{2} = \boxed{\frac{1}{2}}.$$

8 pts (c) $\displaystyle\int_1^4 \frac{1}{(x-3)^2}\,dx$

Solution: This integral is also improper, because the integrand is undefined when $x = 3$, which is between the bounds of integration. We need to split it up into two separate integrals.

$$\begin{aligned}
\int_1^4 \frac{1}{(x-3)^2}\,dx &= \int_1^3 \frac{1}{(x-3)^2}\,dx &&+ \int_3^4 \frac{1}{(x-3)^2}\,dx \\
&= \lim_{b\to 3^-}\int_1^b \frac{1}{(x-3)^2}\,dx &&+ \lim_{a\to 3^+}\int_a^4 \frac{1}{(x-3)^2}\,dx \\
&= \lim_{b\to 3^-}\frac{-1}{(x-3)}\Bigg|_1^b &&+ \lim_{a\to 3^+}\frac{-1}{(x-3)}\Bigg|_a^4 \\
&= \lim_{b\to 3^-}\frac{-1}{(b-3)} - \frac{1}{2} &&+ (-1) - \lim_{a\to 3^+}\frac{-1}{(a-3)}
\end{aligned}$$

Neither limit converges, and so $\boxed{\text{the integral diverges}}$.

12 pts 2. Compute the definite integral $\displaystyle\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$.

Solution: We make the substitution $x = 2\sin\theta$ so $dx = 2\cos\theta d\theta$. When $x = 0$, we have $\theta = 0$ and when $x = 2$, $\theta = \pi/2$ (since $\sin(\pi/2) = 1$). After we make this substitution, we will use the identity $4(1-\sin^2\theta) = 4\cos^2\theta$ to get rid of the square root. We have

$$\begin{aligned}
\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int_0^{\pi/2} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,(2\cos\theta\,d\theta) \\
&= 4\int_0^{\pi/2} \frac{2\sin^2\theta\cos\theta}{\sqrt{4\cos^2\theta}}\,d\theta \quad = \quad 4\int_0^{\pi/2} \frac{2\sin^2\theta\cos\theta}{2\cos\theta}\,d\theta \\
&= 4\int_0^{\pi/2} \sin^2\theta\,d\theta \qquad \text{(Now use } \sin^2\theta = \tfrac{1}{2}\big(1-\cos(2\theta)\big)\text{)} \\
&= 4\int_0^{\pi/2} \tfrac{1}{2}\big(1-\cos(2\theta)\big)\,d\theta \\
&= 2\big(\theta - \tfrac{1}{2}\sin(2\theta)\big)\Big|_0^{\pi/2} \\
&= 2\left(\frac{\pi}{2} - \frac{\sin 2\pi}{2} - (0-0)\right) = \boxed{\pi}.
\end{aligned}$$

12 pts 3. Compute the indefinite integral $\displaystyle\int \frac{5x^2-x+3}{x(x^2+1)}\,dx$.

Solution: First we use partial fractions to decompose $\frac{5x^2-x+3}{x(x^2+1)}$ into a sum of rational functions. We write $\dfrac{5x^2-x+3}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$, and clearing the denominators gives

$$5x^2-x+3 = A(x^2+1) + (Bx+C)(x)\,.$$

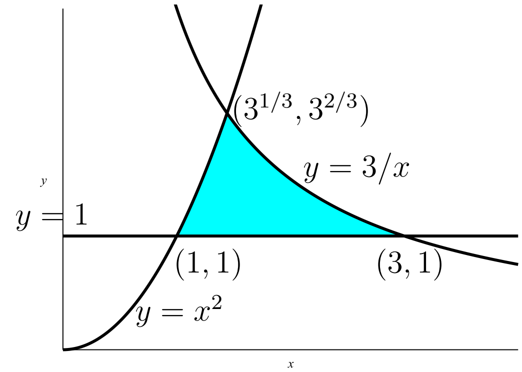
Let $x=0$ to obtain $3 = A$.
Let $x = 1$ (and use the fact that $A = 3$) to get $7 = 3\cdot 2 + (B+C)$, so $1 = B+C$.
Similarly, let $x=-1$ to obtain $9 = 3\cdot 2 + (-B+C)(-1)$, or $3 = B-C$.
Adding these two equations gives $4 = 2B$, and so $B = 2$.
Since $B + C = 1$, this means $C = -1$.

If you prefer, you can multiply out to get $5x^2-x+3 = (A+B)x^2 + Cx + A$,
so $A = 3$, $C = -1$, and $A + B = 5$, that is, $B = 2$.

$$\begin{aligned}
\int \frac{5x^2-x+3}{x(x^2+1)}\,dx = \int \frac{3}{x} + \frac{2x-1}{x^2+1}\,dx &= \int \frac{3}{x}\,dx + \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx \\
&= \boxed{3\ln|x| + 2\ln|x^2+1| - \arctan x + C}.
\end{aligned}$$

12 pts

4. Let $f(x) = x^2$ and $g(x) = 3/x$. Find the area of the region in the first quadrant lying below the graphs of both $f(x)$ and $g(x)$ and above the line $y = 1$.



Solution: First, we need to determine the three points where the curves intersect (the vertices of the curved triangular shape).

The parabola $y = x^2$ intersects the line $y = 1$ at the point $(1, 1)$,
and $y = 3/x$ intersects it at $(3, 1)$.
The third point is the intersection of $y = x^2$ and $y = 3/x$, which we find by solving $x^2 = 3/x$, that is, $x^3 = 3$. This gives us the point $(3^{1/3}, 3^{2/3})$.

Calculating the area means we need to compute an integral. It is most straightforward if we integrate with respect to y, because in this case there will be only a single region.

The rightmost curve is $y = 3/x$, which we can write as $x = 3/y$. The leftmost curve is $y = x^2$ which we write as $x = y^{1/2}$, and since we are integrating with respect to y, the bounds go from 1 to $3^{2/3}$. This gives us

$$\int_1^{3^{2/3}} (3/y - y^{1/2})\, dy = 3\ln(y) - \tfrac{2}{3}y^{3/2}\Big|_1^{3^{2/3}} = \big(3\ln(1) - \tfrac{2}{3}\big) - \big(3\ln(3^{2/3}) - \tfrac{2}{3}(3^{2/3})^{3/2}\big)$$
$$= (0 - \tfrac{2}{3}) - (2\ln(3) - \tfrac{2}{3}\cdot 3) = \boxed{2\ln 3 - \frac{4}{3}}.$$

If instead you insist on integrating with respect to x, observe that we must split the region into two pieces, one where the curve on top is $y = x^2$, and the other where the top curve is $y = 3/x$. This gives the calculation

$$\int_1^{3^{1/3}} (x^2 - 1)\, dx + \int_{3^{1/3}}^3 (3/x - 1)\, dx = \tfrac{1}{3}x^3 - x\Big|_1^{3^{1/3}} + \ 3\ln x - x\Big|_{3^{1/3}}^3$$
$$= \big((\tfrac{(3^{1/3})^3}{3} - 3^{1/3}) - (\tfrac{1}{3} - 1)\big) + \big((3\ln(3) - 3) - (3\ln(3^{1/3}) - 3^{1/3}\big)$$
$$= (1 - 3^{1/3} + \tfrac{2}{3}) + (3\ln(3) - 3 - \ \ln 3 + 3^{1/3})$$
$$= \boxed{2\ln 3 - \frac{4}{3}}.$$

12 pts 5. Use the trapezoid rule with 4 intervals to calculate an approximation to $\ln(3)$.

It is not necessary to add fractions, just write them as a sum.

If you don't remember the trapezoid rule, you may instead use a Riemann sum with 4 rectangles evaluated on the right, but lose 4 points for doing so.

Solution: The first step is to write an integral that represents $\ln 3$. For some reason that eludes me, this seemed to be the hardest part of the problem for many people.

Remember that $\ln x$ is the derivative of $\frac{1}{x}$, and that $\ln 1 = 0$. Consequently

$$\int_1^3 \frac{1}{x}\, dx = \ln 3 - \ln 1 = \ln 3\,.$$

A few people used $\int_2^6 \frac{dx}{x} = \ln 6 - \ln 2 = \ln 3$; that's fine, too. Using $\int_0^1 \ln 3 \; dx$ is kinda cheating; you are approximating $\ln 3$ by the value of $\ln 3$! Note that trying to use $\int_1^3 \ln x \; dx$ is both wrong and wrong-headed: the value of the integral is $3\ln 3 - 2$, and the resulting approximation requires you to know $\ln 3$ to compute it.

Once you have the proper integral in mind, we just apply the trapezoid rule. Since we are using 4 intervals between 1 and 2, each of of length $1/2$, so the 5 points where we evaluate the function are 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and 3. Then we have

$$\begin{aligned}
\ln 3 &\approx \frac{2}{2\cdot 4}\left(f(1) + 2f(\tfrac{3}{2}) + 2f(2) + 2f(\tfrac{5}{2}) + f(3)\right) \\
&= \frac{1}{4}\left(1 + 2(\tfrac{2}{3}) + 2(\tfrac{1}{2}) + 2(\tfrac{2}{5}) + \tfrac{1}{3}\right) \\
&= \frac{1}{4}\left(1 + \tfrac{4}{3} + 1 + \tfrac{4}{5} + \tfrac{1}{3}\right) \\
&= \boxed{\frac{67}{60}}\,.
\end{aligned}$$

Since $67/60 \approx 1.117$ and $\ln 3 \approx 1.0986$, this isn't too bad. Of course, Simpson's rule gives an approximation more than ten times better, with no more calculation. (The points are the same, but just change the weight at $3/2$ and $5/2$ to a 4, and divide the sum by 6 instead of 4. Even Simpson's rule with 2 intervals (which is $\frac{1}{3}(1 + 2 + 1/3) = 10/9$) gives a slightly better approximation. But there is no Simpson's rule on this test.