MAT126 Fall 2009
Practice Midterm II

The actual midterm will consist of six problems

Problem 1 If the function $g(x)$ is given by

$$g(x) = \int\limits_{2x}^{x^2} t \ln t dt,$$

compute the derivative $g'(x)$

a) by using the Fundamental Theorem of Calculus to differentiate the integral

b) by using the Evaluation Theorem to first evaluate $g(x)$ explicitly, and then differentiating.

Solution:

a) We note that $g(x)$ has the form similar to the fundamental theorem of calculus, but the limits of integration are not as required to apply the theorem. To apply the fundamental theorem of calculus, we denote

$$F(x) = \int_0^x t \ln t dt,$$

so that then $g(x) = F(x^2) - F(2x)$ (recall that the integral of a function from $2x$ to x^2 is equal to the difference of the integrals from 0 to x^2 and from 0 to $2x$).

By the fundamental theorem of calculus we have $F'(x) = x \ln x$, and thus by applying the chain rule we have

$$g'(x) = F'(x^2) \cdot 2x - F'(2x) \cdot 2 = x^2 \ln(x^2) \cdot 2x - 2x \ln(2x) \cdot 2 = 4x^3 \ln(x) - 4x \ln(2x),$$

where we have used $\ln(x^2) = 2 \ln x$ for the last identity.

b) We are going to compute the integral

$$\int\limits_{2x}^{x^2} t \ln t dt$$

by parts. Since we want to get the $\ln t$ differentiated, we will use $u = \ln t$, which forces us to have $dv = tdt$, so that $du = \frac{dt}{t}$ and $v = \frac{t^2}{2}$. We thus obtain

$$g(x) = \int\limits_{2x}^{x^2} t \ln t dt = \left[\frac{t^2 \ln t}{2} \right]_{2x}^{x^2} - \int\limits_{2x}^{x^2} \frac{t^2 dt}{2t}$$

$$= \left[\frac{t^2 \ln t}{2}\right]_{2x}^{x^2} - \int_{2x}^{x^2} \frac{tdt}{2} = \left[\frac{t^2 \ln t}{2} - \frac{t^2}{4}\right]_{2x}^{x^2} = \frac{x^4 \ln(x^2) - 4x^2 \ln(2x)}{2} - \frac{x^4 - 4x^2}{4}$$

$$= x^4 \ln(x) - 2x^2 \ln(2x) + x^2 - \frac{x^4}{4},$$

where we have again used $\ln(x^2) = 2\ln x$. We now use the chain and product rules to compute the derivative

$$g'(x) = 4x^3 \ln(x) + \frac{x^4}{x} - 4x \ln(2x) - \frac{2x^2 \cdot 2}{2x} + 2x - x^3 =$$

$$4x^3 \ln(x) + x^3 - 4x \ln(2x) - 2x + 2x - x^3 = 4x^3 \ln(x) - 4x \ln(2x)$$

matching the answer in part a).

Problem 2 Evaluate the following definite integrals:

1. $$\int_0^2 x^2\sqrt{4-x^2}dx$$

2. $$\int_1^{e^\pi} \frac{\cos(\ln x)\sin^2(\ln x)}{x}dx$$

3. $$\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^2}dx$$

Solution:

1. In this case a trigonometric substitution is called for. We substitute $x = 2\sin\theta$, in which case

   $$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta \quad \text{and} \quad dx = d(\sin\theta) = \cos\theta d\theta$$

   so we get (don't forget to change the limits of integration, using $\theta = \arcsin(x/2)$!)

   $$\int_0^2 x^2\sqrt{4-x^2}dx = \int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta$$

   Using the half-angle formulas we have

   $$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)); \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

   and thus the above integral becomes

   $$\int_0^{\pi/2} \frac{1}{4}(1-\cos(2\theta))(1+\cos(2\theta))d\theta = \frac{1}{4}\int_0^{\pi/2}(1-\cos^2(2\theta))d\theta$$

   $$\frac{1}{4}\int_0^{\pi/2}\sin^2(2\theta)d\theta = \frac{1}{4}\int_0^{\pi/2}\frac{1-\cos(4\theta)}{2}d\theta =$$

   $$\frac{1}{8}\left[x - \frac{\sin(4\theta)}{4}\right]_0^{\pi/2} = \frac{1}{8}\left(\pi/2 - 0 - \frac{\sin(2\pi)-\sin(0)}{4}\right) = \frac{\pi}{16}$$

   where we computed $\int\cos(4\theta)d\theta = \frac{1}{4}\sin(4\theta)+c$ in the last line.

2. Notice that in this case both sin and cos are evaluated at $\ln x$. This essentially means that our only chance to deal with the integral is to make $u = \ln x$, so that we could potentially handle these integrals. We do this substitution, noting that $du = dx/x$, so that we get (don't forget to change the limits of integration!)

$$\int_1^{e^\pi} \frac{\cos(\ln x)\sin^2(\ln x)}{x} dx = \int_0^\pi \cos u \sin^2 u du$$

(for the limits of integration, note that for $x = 1$ we have $u = \ln 1 = 0$, and $u(e^\pi) = \ln(e^\pi) = \pi$).

Now we have a trigonometric integral with both sin and cos, and it is thus natural to substitute $t = \sin u$, so that $dt = \cos u du$. It seems we should thus get (changing the limits of integration again!)

$$\int_0^\pi \cos u \sin^2 u du = \int_{\sin 0}^{\sin(\pi)} t^2 dt = \int_0^0 t^2 dt = 0$$

However, this is cheating, as for substitution rule we really need to make sure that u is an increasing everywhere (or decreasing everywhere function). So we should really compute the indefinite integral first, in terms of x. The answer is, however, the same - try it!

3. Here again we see that sin is evaluated at some complicated point, so our only chance is to substitute $u = \frac{1}{x}$, so that $du = -\frac{1}{x^2}dx$. We thus get (careful with the limits of integration!)

$$\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^2} dx = \int_\pi^{\pi/2} -\sin u du = \cos u|_\pi^{\pi/2} = 1$$

Problem 3 Evaluate the following indefinite integral using integration by parts:

1.
$$\int \arcsin(x)dx$$

2.
$$\int \sqrt{x}\ln^2(x)dx$$

Solution:

1. In the first case we really do not have any choice for which parts to take: we must set $u = \arcsin x$, in which case we have simply $dv = dx$ (this is what is left in the formula, we must have $\int \arcsin x dx = \int u dv$), so that $v = x$ and $du = \frac{1}{\sqrt{1-x^2}}$. We thus obtain
$$\int \arcsin(x)dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx.$$
For the second integral we substitute $t = 1 - x^2$ (otherwise we would not be able to handle the square root), so that $dt = -2xdx$. So
$$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{dt}{-2t} = -\frac{1}{2}\int t^{-1/2}dt = -t^{1/2} + c = -\sqrt{1-x^2} + c,$$
where we first substituted and then returned to the initial variable x. The final answer is thus
$$\int \arcsin(x)dx = x \arcsin x + \sqrt{1-x^2} + c.$$

2. Here we recall that the derivative of $\ln x$ is $1/x$, and thus we want to get $\ln x$ differentiated. We thus set $u = (\ln x)^2$, so that $dv = \sqrt{x}dx = x^{1/2}dx$ (if we chose say $u = \ln x$, we would not be able to compute v). Then by the chain rule for differentiation $du = \frac{2\ln x}{x}dx$ and $v = \frac{2}{3}x^{3/2}$. Thus
$$\int \sqrt{x}\ln^2 x dx = \frac{2}{3}x^{3/2}\ln^2 x - \int \frac{2\ln x}{x}\frac{2}{3}x^{3/2}dx = \frac{2}{3}x^{3/2}\ln^2 x - \frac{4}{3}\int x^{1/2}\ln x dx.$$
To deal with the integral that appeared here, we will need to again integrate by parts, using similarly to the above $u = \ln x$ and $dv = x^{1/2}dx$, so that $v = \frac{2}{3}x^{3/2}$ and $du = \frac{dx}{x}$. So
$$\int x^{1/2}\ln x dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\frac{dx}{x} =$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + c$$

and the final answer is then

$$\int \sqrt{x}\ln^2 x dx = \frac{2}{3}x^{3/2}\ln^2 x - \frac{8}{9}x^{3/2}\ln x + \frac{16}{27}x^{3/2} + c.$$

Problem 4 Evaluate
$$\lim_{n\to\infty}\int_0^{2\pi} x\sin(nx)dx$$

Solution:

We begin by evaluating the integral in its general form, for any value of n. To do this, we first make the following substitution

$$t = nx, dt = ndx,$$

so that we have
$$x = \frac{1}{n}t, dx = \frac{1}{n}dt$$
Applying these substitutions to our original integral, we have (remember to change the limits of integration!)
$$\int_0^{2\pi} x\sin(nx)dx = \int_0^{2\pi} \frac{1}{n}t\sin(t)\frac{1}{n}dt = \frac{1}{n^2}\int_0^{2n\pi} t\sin(t)dt$$
To continue doing the integral, we do an integration by parts with

$$u = t, \quad dv = \sin(t)dt$$

$$du = dt, \quad v = -\cos(t)$$

Performing integration by parts
$$\int_0^{2n\pi} x\sin(nx)dx = \frac{1}{n^2}\int_0^{2n\pi} t\sin(t)dt$$
$$= \frac{1}{n^2}\left[-t\cos(t)\right]_0^{2n\pi} - \frac{1}{n^2}\int_0^{2n\pi} -\cos(t)dt = \frac{1}{n^2}\left[-t\cos(t)\right]_0^{2\pi} + \frac{1}{n^2}\int_0^{2n\pi}\cos(t)dt$$
$$= \frac{1}{n^2}\left[-t\cos(t)\right]_0^{2n\pi} + \frac{1}{n^2}\left[\sin(t)\right]_0^{2n\pi}$$
$$= \frac{1}{n^2}\left[(-2n\pi\cos(2n\pi)) - (-n0\cos(0))\right] + \frac{1}{n^2}\left[(\sin(2n\pi)) - (\sin(0))\right]$$
$$= \frac{1}{n^2}\left[(-2n\pi) - (0)\right] + \frac{1}{n^2}\left[(0) - (0)\right] = -\frac{2\pi}{n}$$
Taking the limit as n goes to infinity, the answer is
$$\lim_{n\to\infty}\int_0^{2\pi} x\sin(nx)dx = \lim_{n\to\infty} -\frac{2\pi}{n} = 0.$$

Problem 5 Decompose a rational function into partial fractions

$$\frac{x^3 - 6x}{x^2 + 4x + 4}$$

Solution:

Since the degree of the polynomial on top is more than on the bottom, we begin by performing long division: multiplying $x^2 + 4x + 4$ by x yields

$$x^3 + 4x^2 + 4x.$$

Subtracting this from

$$x^3 - 6x$$

yields

$$-4x^2 - 10x$$

multiplying $x^2 + 4x + 4$ by -4 yields

$$-4x^2 - 16x - 16$$

subtracting this from $-4x^2 - 10x$ yields

$$6x + 16$$

So we know that

$$\frac{x^3 - 6x}{x^2 + 4x + 4} = x - 4 + \frac{6x + 16}{x^2 + 4x + 4}$$

We note that $x^2 + 4x + 4 = (x + 2)^2$, and thus we cannot expect to have

$$\frac{6x + 16}{x^2 + 4x + 4} = \frac{A}{x + 2}.$$

Instead we would expect to have

$$\frac{6x + 16}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}.$$

When we clear the denominators by multiplying by $(x + 2)^2$, we end up with the equation

$$6x + 16 = A(x + 2) + B$$

which must be satisfied for all x. Thus from the coefficient of x we see that $A = 6$, and then $B = 16 - 2A = 4$.

So our answer is

$$\frac{x^3 - 6x}{x^2 + 4x + 4} = x - 4 + \frac{6}{x+2} + \frac{4}{(x+2)^2}$$

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Problem 6 Evaluate the integral

$$\int_0^1 \frac{x+1}{x^2+9} dx$$

Solution: We split the integral into two summands, writing

$$\int_0^1 \frac{x+1}{x^2+9} dx = \int_0^1 \frac{x}{x^2+9} dx + \int_0^1 \frac{1}{x^2+9} dx.$$

For the first summand we are going to substitute $u = x^2 + 9$, so that $du = 2xdx$. We then get

$$\int_0^1 \frac{x}{x^2+9} dx = \frac{1}{2} \int_{u(0)}^{u(1)} \frac{du}{u} = \frac{1}{2} \int_9^{10} \frac{du}{u} = \frac{1}{2} \ln u \Big|_9^{10} = \frac{1}{2}(\ln(10) - \ln 9).$$

For the second summand we recognize this as something similar to the arctan. Indeed, to bring it to the standard form we will need to substitute $x = 3u$, so that $dx = 3du$ (and we have $u = x/3$). The integral is then

$$\int_0^1 \frac{1}{x^2+9} dx = \int_{u(0)}^{u(1)} \frac{3du}{(3u)^2+9} = \int_0^{1/3} \frac{du}{3(u^2+1)}$$

$$= \frac{1}{3} \arctan u \Big|_0^{1/3} = \frac{1}{3}(\arctan(1/3) - \arctan 0) = \frac{\arctan(1/3)}{3}$$

since $\tan 0 = 0$, and thus $\arctan 0 = 0$. The final answer is thus

$$\frac{\ln(10) - \ln 9}{2} - \frac{\arctan(1/3)}{3}.$$

Problem 7 Can the midpoint approximation to the integral

$$\int_1^2 \frac{1}{x^2}$$

with $n = 100$ be equal to

- $\frac{1}{7}$.
- $\frac{1}{5}$

To get a full credit you need to justify your answer

Solution:

1. Calculating the integral $\int_1^2 \frac{1}{x^2}dx$ we get:

$$\int_1^2 \frac{1}{x^2}dx = -\left.\frac{1}{x}\right|_1^2 = -\left(\frac{1}{2} - \frac{1}{1}\right) = \frac{1}{2}$$

2. Then, we can use the formula for the error bound of the Midpoint rule:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$.

Note that in our case $f(x) = \frac{1}{x^2}$. Therefore $f'(x) = \frac{-2}{x^3}$ and $f''(x) = \frac{6}{x^4}$. Since $1 \leq x \leq 2$, we have that $\frac{6}{2^4} \leq \frac{6}{x^4} \leq 6$. So, we can use $K = 6$ for the error bound of the midpoint approximation.

$$|E_M| \leq \frac{6 \cdot (2-1)^3}{24 \cdot 100^2} = \frac{1}{40000}$$

3. By all the above we have that:

$$\frac{1}{2} - \frac{1}{40000} \leq M_{100} \leq \frac{1}{2} + \frac{1}{40000}$$

which means that the approximation M_{100} is very close to the value of the integral, which is $\frac{1}{2}$. Therefore, M_{100} cannot be equal to $\frac{1}{5}, \frac{1}{7}$ which are both much less than $\frac{1}{2} - \frac{1}{40000}$.