Stony Brook University

NAME: Midterm Solutions

RECITATION #: ____

MAT 126 – Spring 2016 – Midterm 1

February 23, 2016

INSTRUCTIONS – PLEASE READ

Please turn off your cell phone and put it away.

➩ Please write your name and your section number right now.

➩ This is a closed book exam. You are NOT allowed to use a calculator or any other electronic device or aid.

➩ The midterm has 6 problems worth a total of 100 points. Look over your test packet as soon as the exam begins. If you find any missing pages or problems please ask a proctor for another test booklet.

➩ Show your work. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly. You do not need to simplify your answers unless explicitly instructed to do so.

➩ Academic integrity is expected of all Stony Brook University students at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1.	
2.	
3.	
4.	
5.	
6.	
Total	

LEC 01	MWF	10:00-10:53am	Joseph Adams
R01	F	1:00-1:53pm	Jaroslaw Jaracz
R02	Tu	4:00-4:53pm	Charles Cifarelli
R03	Tu	1:00-1:53pm	Jaroslaw Jaracz
R04	Th	8:30-9:23am	Alaa Abd-El-Hafez
R05	M	1:00-1:53pm	Thomas Rico
R06	M	9:00-9:53am	Zhuang Tao
R07	W	11:00-11:53am	Dyi-Shing Ou
LEC 02	TuTh	2:30-3:50pm	Raluca Tanase*
R08	Tu	4:00-4:53pm	Gaurish Telang
R09	Tu	1:00-1:53pm	Yuan Gao
R10	Th	1:00-1:53pm	Alaa Abd-El-Hafez
R11	F	1:00-1:53pm	Ruijie Yang
R12	W	12:00-12:53pm	Christopher Ianzano
R13	M	10:00-10:53am	Zhuang Tao
R14	M	12:00-12:53pm	Thomas Rico
LEC 03	MW	4:00-5:20pm	David Kahn
R15	W	9:00-9:53am	Ruijie Yang
R16	Tu	10:00-10:53am	Nicholas Valente
R17	W	10:00-10:53am	Nicholas Valente
R18	Th	4:00-4:53pm	Gaurish Telang
R31	W	5:30-6:23pm	Mariangela Ferraro
R32	M	5:30-6:23pm	Charles Cifarelli
R33	Tu	1:00-1:53pm	Yu Zeng

Problem 1. (17 points)

a) Find the function $f(y)$ such that $f'(y) = e^y + \cos(y)$ and $f(0) = 10$.

$$f(y) = \int f'(y)\,dy = \int (e^y + \cos(y))\,dy = e^y + \sin(y) + c$$

$$f(0) = 10 \Rightarrow e^0 + \sin(0) + c = 10 \Rightarrow c = 9$$

$$f(y) = e^y + \sin(y) + 9$$

b) Let $h(y) = \int\limits_2^y \sqrt{t^3+1}\,dt$. Compute $h(2)$ and $h'(2)$.

$$h(2) = \int_2^2 \sqrt{t^3+1}\,dt = 0$$

$$h'(y) = \frac{d}{dy}\left(\int_2^y \sqrt{t^3+1}\right)dt = \sqrt{y^3+1}$$

$$h'(2) = \sqrt{2^3+1} = 3$$

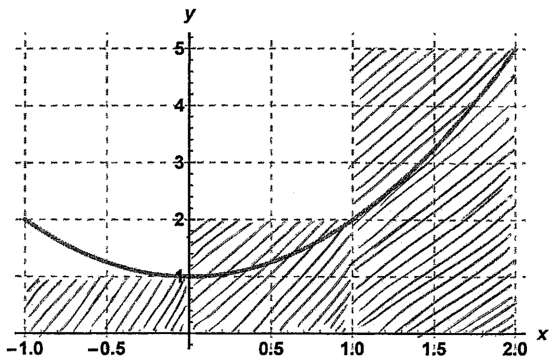
c) Suppose that g is an integrable functions on $[0,3]$, such that $\int_1^3 g(x)dx = 6$, $\int_1^0 g(x)dx = -5$ and $\int_2^3 g(x)dx = 1$. Compute $\int_0^2(-2g(x)+5)dx$.

$$\int_0^2 g(x)\,dx = \int_0^1 g(x)\,dx + \int_1^2 g(x)\,dx = -\int_1^0 g(x)\,dx + \int_1^3 g(x)\,dx - \int_2^3 g(x)dx$$

$$= -(-5) + 6 - 1 = 10$$

$$\int_0^2 (-2g(x)+5)\,dx = -2\int_0^2 g(x)dx + \int_0^2 5dx = -20 + 10 = -10$$

Problem 2. (20 points) Consider the function $f(x) = x^2 + 1$, defined on the interval $[-1, 2]$.



a) Approximate the area between the graph of f and the x-axis using 3 right hand rectangles with equal widths.

$$\Delta x = \frac{2-(-1)}{3} = \frac{3}{3} = 1$$

$$A \simeq \left(f(0) + f(1) + f(2)\right)\Delta x = (1+2+5)\cdot 1 = 8$$

b) Write a formula for a Riemann Sum with n right hand rectangles.

$$R_n = \sum_{k=1}^{n} f(-1 + k\Delta x)\,\Delta x \qquad \text{where } \Delta x = \frac{3}{n}$$

$$R_n = \sum_{k=1}^{n} \left(\left(-1 + 3\frac{k}{n}\right)^2 + 1\right)\cdot \frac{3}{n}$$

c) Evaluate the limit of the Riemann Sum from part (b) as $n \to \infty$, either using integrals, or by direct computation, using the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Sol 1

$$\lim_{n\to\infty} R_n = \int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} (x^2+1)\,dx = \frac{x^3}{3} + x \Big|_{-1}^{2}$$

$$= \left(\frac{8}{3} + 2\right) - \left(\frac{-1}{3} - 1\right) = \frac{9}{3} + 3 = 6$$

<u>Sol 2</u>

$$R_n = \sum_{k=1}^{n} \frac{6}{n} + \sum_{K=1}^{n} \frac{27K^2}{n^3} - \sum_{K=1}^{n} 18\frac{K}{n^2}$$

$$R_n = 6\cdot\frac{\cancel{n}}{\cancel{n}} + \frac{27}{n^3}\sum_{K=1}^{n} K^2 - \frac{18}{n^2}\sum_{K=1}^{n} K$$

$$= 6 + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{18}{n^2}\,\frac{n(n+1)}{2}$$

$$= 6 + \frac{9(n+1)(2n+1)}{2n^2} - 9\,\frac{n+1}{n}$$

$$= 6 + 9\left(\frac{2n^2+3n+1}{2n^2}\right) - 9\left(1+\frac{1}{n}\right)$$

$$= 6 + 9\left(1+\frac{3}{2n}+\frac{1}{2n^2}\right) - 9\left(1+\frac{1}{n}\right) = 6 + 9\left(\frac{1}{2n}+\frac{1}{2n^2}\right)$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} 6 + 9\left(\underbrace{\frac{1}{2n}}_{0} + \underbrace{\frac{1}{2n^2}}_{0}\right) = 6.$$

Problem 3. (21 points) Evaluate the following expressions:

a) $\int_{2}^{5} (6x^2 + 4x - 1)\, dx$ $= \frac{6x^3}{3} + \frac{4x^2}{2} - x \Big|_2^5 = 2x^3 + 2x^2 - x \Big|_2^5$

$= (250 + 50 - 5) - (16 + 8 - 2)$

$= 295 - 22 = 273$

b) $\int \frac{\sin^2(x)}{\sec(x) - \sec(x)\cos^2(x)}\, dx$ $= \int \frac{\sin^2 x}{\sec x\,(1 - \cos^2 x)}\, dx =$

$= \int \frac{\sin^2 x}{\sec x \cdot \sin^2 x}\, dx = \int \frac{1}{\sec x}\, dx = \int \cos x$

$= \sin x + C$

c) $\frac{d}{dx}\left(\int_{2x}^{x^5} \tan(\ln(t^2))\, dt\right)$ $= \tan(\ln((x^5)^2)) \cdot (x^5)' - \tan(\ln((2x)^2)) \cdot (2x)'$

$= \tan(\ln(x^{10})) \cdot 5x^4 - \tan(\ln(4x^2)) \cdot 2$

Problem 4. (14 points) Calculate the following integrals, using the appropriate substitution:

a) $\int \frac{1}{x(\ln x)^2}\,dx$ $= \int \frac{1}{u^2}\,du = \int u^{-2}\,du =$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \frac{u^{-2+1}}{-2+1} + c = -\frac{1}{u} + c$

$= -\frac{1}{\ln x} + c$

b) $\int_0^{\sqrt[3]{\pi}} x^2 \sin(x^3)\,dx$ $= \int_0^{\pi} \frac{1}{3} \sin(u)\,du = -\frac{\cos(u)}{3}\Big|_0^{\pi}$

$u = x^3$

$du = 3x^2\,dx$

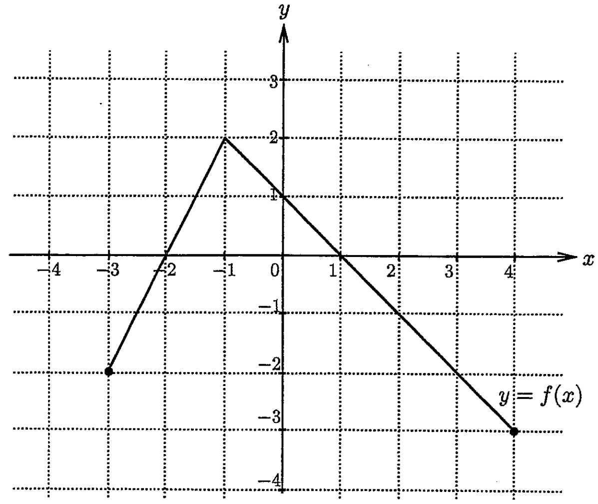
$x = 0 \Rightarrow u = 0$

$x = \sqrt[3]{\pi} \Rightarrow u = \pi$

$= \frac{(-\cos(\pi)) - (-\cos 0)}{3}$

$= \frac{-(-1) - (-1)}{3} = \frac{2}{3}$

Problem 5. (20 points) Consider the function $f(x)$ graphed below:



Now define a **new** function $F(x) = \int_{-3}^{x} f(t)\,dt$ on the interval $[-3, 4]$.

a) Compute $F(-3)$, $F(1)$ and $F(4)$.

$F(-3) = \int_{-3}^{-3} f(t)\,dt = 0$.

$F(1) = \frac{2\cdot 3}{2} - \frac{2\cdot 1}{2} = 2$

$F(2) = F(1) + \int_{1}^{4} f(t)\,dt$

$= 2 - \frac{3\cdot 3}{2} = -\frac{5}{2}$

b) Where is F increasing? Where is F decreasing?

$F'(x) = \frac{d}{dx}\left(\int_{-3}^{x} f(t)\,dt\right) = f(x)$. by FTC

F is increasing when f is positive, that is on $[-2, 1]$.

F is decreasing when $f(x) < 0$, that is on $[-3, -2] \cup [1, 4]$.

c) Where is F concave-up? Where is F concave-down?

$F''(x) = f'(x)$

F is concave-up when $F''(x) > 0$, that is when $f'(x) > 0$, or equivalently when f is increasing (its slope is positive)

So F is concave-up on $[-3, -1]$ and concave-down on $[-1, 4]$.

Problem 6. (8 points) Determine whether the following statements are true or false. Circle your response and give a brief explanation (a reason why it's true or an example where it fails).

a) TRUE | FALSE Suppose that f and g are two integrable functions on $[0,1]$. Then

$$\int_0^1 f(x)g(x)dx = \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

Take for example $f(x) = x$ and $g(x) = x$

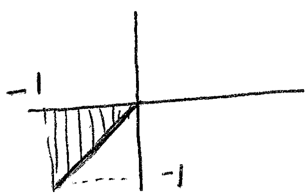
$$\int_0^1 f(x)g(x)\,dx = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\int_0^1 f(x)dx \int_0^1 g(x)\,dx = \left(\int_0^1 x\,dx\right)^2 = \left(\frac{x^2}{2}\Big|_0^1\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{1}{3} \neq \frac{1}{4}$$

b) TRUE | FALSE Let f be an integrable functions on $[a,b]$. The definite integral $\int_a^b f(x)dx$ represents the area of the region enclosed between the graph of the function and the x-axis.

Assume f is negative. Take for example $f(x) = x$ on $[-1,0]$.



Then $\int_{-1}^0 f(x)\,dx = \int_{-1}^0 x\,dx = \frac{x^2}{2}\Big|_{-1}^0 = -\frac{1}{2}$

However, the area of the region between the graph of f and the x-axis is positive

$$A = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

So in this case $\int_{-1}^0 f(x)\,dx = -A$