SOLUTIONS, SUMMER 16

1. (20 points) Write the following as a definite integral and then solve:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{16-(x_i^*)^2}\,\Delta x$$

where $x_i^* = 0 + \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$.

SINCE $x_i = \frac{4}{n} i$, $a = 0$, $b = 4$.

(THERE ARE OTHER POSSIBILITIES, BUT THIS IS THE OBVIOUS ONE)

AND $f(x) = \sqrt{16 - x^2}$

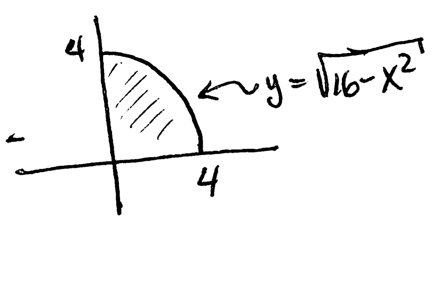
THUS THE INTEGRAL IS

$$\int_0^4 \sqrt{16-x^2}\,dx.$$

YOU CAN DO THIS INTEGRAL ~~AS A~~ BY A TRIG SUBSTITUTION ($x = 4\sin\theta$, $dx = 4\cos\theta\, d\theta$, $0 \le \theta \le \pi/2$) TO GET

$$\int_0^{\pi/2} 16 \cos^2\theta\, d\theta$$

BUT IT IS EASIER TO NOTICE THIS IS THE QUARTER-CIRCLE $y^2 + x^2 = 16$ OF RADIUS 4, SO THE INTEGRAL IS JUST



$$\frac{1}{4}(16\pi) = \boxed{4\pi.}$$

SOLUTIONS SUMMER '16

2. (20 points) Compute the following:

$$\frac{d}{dx}\left(\int_{x^2}^{\frac{1}{x}} e^{(v^3+7)^4}\, dv\right)$$

BY THE FUND. THM, WE GET

$$-\frac{1}{x^2} e^{\left(\frac{1}{x^3}+7\right)^4} - 2x\, e^{(x^6+7)^4}$$

(THAT IS, PLUG IN ENDPOINTS, MULTIPLY BY DERIVATIVE)

3. (20 points) Solve the following integral:

$$\int w^4 e^{-w^5}(e^{-w^5}+7)^3\, dw$$

MAKE THE SUBSTITUTION

$u = e^{-w^5}+7$, $du = -5w^4 e^{-w^5}\, dw$

THEN THE INTEGRAL BECOMES

$$-\frac{1}{5}\int u^3\, du = -\frac{1}{5}\cdot\frac{u^4}{4} + C$$

$$\boxed{= -\frac{(e^{-w^5}+7)}{20} + C}$$

SOL'NS SUMMER '16

4. (20 points) Compute the following:

$$\int \frac{\cos\theta}{(3+\sin\theta)^2(4+\sin\theta)}\,d\theta$$

(HINT: Let $u = \sin(\theta)$ and proceed from there)

AS SUGGESTED, $u = \sin\theta$
$du = \cos\theta \, d\theta$

AND WE GET $\int \frac{du}{(3+u)^2(4+u)}$. NOW USE PARTIAL FRACTIONS.

$$\frac{1}{(3+u)^2(4+u)} = \frac{Au+B}{(3+u)^2} + \frac{C}{4+u}$$

OR $1 = (Au+B)(4+u) + C(3+u)^2$

IF $u=-4$, WE GET $1 = 0 + C(-1)^2$, SO $\boxed{C=1}$

IF $u=0$, WE GET $1 = 4B + 9C = 4B+9$, SO $-8 = 4B$, $\boxed{B=-2}$

LET $u=-3$ TO GET $1 = (-3A-2)(1) + 0$

SO $3 = -3A$ AND $\boxed{A=-1}$

THE INTEGRAL BECOMES

$$\int \left(\frac{-u-2}{(3+u)^2} + \frac{1}{4+u}\right) du = \int \frac{-(u+3)}{(u+3)^2} + \frac{1}{(u+3)^2} + \frac{1}{4+u}\,du$$

$$= -\ln|u+3| - \frac{1}{(u+3)} + \ln|u+4| + C$$

$$= \boxed{-\ln|\sin\theta+3| - \frac{1}{\sin\theta+3} + \ln|\sin\theta+4| + C}$$

SOL'NS SUMMER '16

5. (20 points) Please state whether the following integrals are convergent or divergent. If convergent, state the value for which the integral ultimately converges to. You must justify your answers!

(a) $\int_1^\infty z^7 \ln(z)dz$ $= \lim_{M\to\infty} \int_1^M z^7 \ln z\, dz$

$u = \ln z \quad dv = z^7 dz$

$du = \frac{dz}{z} \quad v = \frac{1}{8}z^8$

$$= \lim_{M\to\infty} \left(\frac{1}{8}z^8 \ln z \Big|_1^M - \frac{1}{8}\int_1^M z^7 dz \right)$$

$$= \lim_{M\to\infty} \frac{1}{8} z^8 \ln z - \frac{1}{64} z^8 \Big|_1^M$$

$$= \left(\lim_{M\to\infty} \frac{1}{8}M^8 \ln M - \frac{1}{64}M^8\right) - \left(0 - \frac{1}{64}\right)$$

$$= +\infty$$

$\boxed{\text{DIVERGES}}$

(b) $\int_0^{\frac{\pi}{4}} \sec^2(2u)\tan^2(2u)\, du$

LET

$w = \tan 2u \quad dw = 2\sec^2(2u)\, du$

$u = 0 \Rightarrow w = 0 \quad u = \pi/4 \Rightarrow w = \tan\frac{\pi}{2} = +\infty.$

$$= \int_0^\infty \frac{1}{2} w^2 dw$$

$$= \lim_{M\to\infty} \int_0^M \frac{1}{2}w^2 dw = \lim_{M\to\infty} \frac{1}{6}w^3 - 0 = +\infty$$

$\boxed{\text{DIVERGES.}}$

6. (20 points) Find the following information for the region bounded by $y = 3 + x^2$, $y = 2 - x^2$, $x = -1$ and $x = 1$:

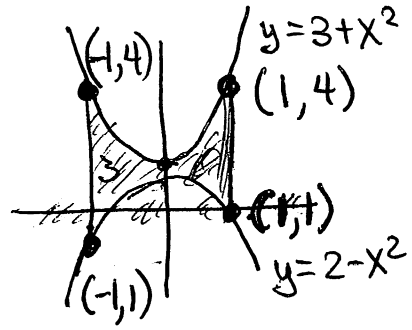
(a) Sketch the region.

(b) Find the area of the region.

(c) Sketch the disk, washer, or shell that we'd obtain if we rotated our region about the line $x = 2$.

(d) Find the volume of the solid from part (c).

a)



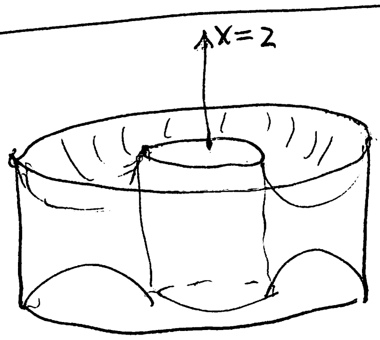
b) $$\text{AREA} = \int_{-1}^{1} (3+x^2)-(2-x^2)\,dx$$

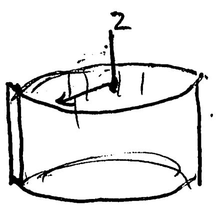
$$= \int_{-1}^{1} (1+2x^2)\,dx$$

$$= x + \frac{2}{3}x^3 \Big|_{-1}^{1}$$

$$= \left(1+\frac{2}{3}\right) - \left(-1-\frac{2}{3}\right) = \boxed{\frac{10}{3}}$$

c)





TYPICAL CYLINDER HAS RADIUS $2-X$

HEIGHT IS $(3+x^2)-(2-x^2) = 1+2x^2$

CIRCUMFERENCE IS $2\pi(2-x)$.

AREA IS $2\pi(2-x)(1+2x^2)$

d) VOLUME IS

$$\int_{-1}^{1} 2\pi(2-x)(1+2x^2)\,dx = 2\pi\int_{-1}^{1} 2-x+4x^2-2x^3\,dx$$

$$= 2\pi\left(2x - \frac{x^2}{2} + \frac{4}{3}x^3 - \frac{x^4}{2}\right)\Big|_{-1}^{1} = 2\pi\left[\left(2-\frac{1}{2}+\frac{4}{3}-\frac{1}{2}\right)-\left(-2-\frac{1}{2}-\frac{4}{3}-\frac{1}{2}\right)\right]$$

$$= 2\pi\left[4+\frac{8}{3}\right] = \boxed{\frac{64\pi}{3}}$$

7. (20 points) Set-up, but DO NOT EVALUATE, the integral that would give us the arc length of

$$x = \ln(\tan(e^{-y})) = f(y)$$

on the interval $2 \leq y \leq 4$.

$$f'(y) = \frac{1}{\tan(e^{-y})} \cdot \sec^2(e^{-y})(-e^{-y})$$

ARC LENGTH $= \int_2^4 \sqrt{1+(f'(y))^2}\, dy$

$$= \int_2^4 \sqrt{1 + \frac{\sec^2(e^{-y})}{\tan^4(e^{-y})} e^{-2y}}\, dy$$

8. (20 points) Find the average of

$$f(x) = \frac{x^2}{\sqrt{x^3+1}}$$

on the interval $[1, 3]$. You may leave your final answer in radical form.

AVERAGE OF $f(x)$ ON $a<x<b$ IS $\frac{1}{b-a} \int_a^b f(x)\, dx$.

SO,

$$\frac{1}{2} \int_1^3 \frac{x^2 dx}{\sqrt{x^3+1}} =$$

$u = x^3+1$ $\quad$ $x=1 \Rightarrow u = 2$

$du = 3x^2 dx$ $\quad$ $x=3 \Rightarrow u = 28$

$$= \frac{1}{6} \int_2^{28} \frac{du}{\sqrt{u}} = \frac{1}{6} \cdot 2\sqrt{u} \Big|_2^{28}$$

$$\boxed{= \frac{1}{3}\left(\sqrt{28} - \sqrt{2}\right)}$$

SOL'NS SUMMER '16

9. (30 points) Please complete the following:

(a) (10 points) A force of $300N$ stretches a spring $30cm$ from its natural length. How much work is done in stretching the spring from $50cm$ from its natural length to $60cm$ from its natural length?

(THIS IS A BIT AMBIGUOUS. I'M GOING TO INTERPRET IT TO MEAN IT TAKES 300N OF FORCE TO HOLD IT AT 30CM = .3M) BY HOOKE'S LAW, THE FORCE IS A CONSTANT TIMES THE STRETCHED DISTANCE, IE $F = .3k = 300N \Rightarrow k = 1000$

THEN THE WORK IS

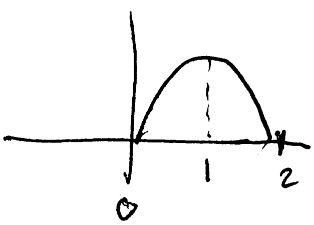
$$\int_{.5}^{.6} 1000\,s\,ds = \frac{1000\,s^2}{2}\Big|_{.5}^{.6}$$

$$= 500(.36 - .25)$$

$$= 55 \text{ JOULES.}$$

(b) (20 points) Find the centroid of the region bounded by $y = 2x - x^2$ and $y = 0$.

[THIS TOPIC WON'T BE ON THE FALL '16 FINAL]



$$A = \text{AREA} = \int_0^2 2x - x^2\,dx = x^2 - \frac{x^3}{3}\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

$$\bar{x} = \frac{3}{4}\int_0^2 x(2x - x^2)\,dx = \frac{3}{4}\left(\frac{2}{3}x^3 - \frac{x^4}{4}\right)\Big|_0^2$$

$$= \frac{3}{4}\left(\frac{2}{3}\cdot 8 - \frac{16}{4}\right) = \frac{3}{4}\cdot\frac{16}{12} = 1$$

$$\bar{y} = \frac{3}{4}\cdot\frac{1}{2}\int_0^2 (2x - x^2)^2 - 0^2\,dx = \frac{3}{8}\int_0^2 (4x^2 - 8x^3 + x^4)\,dx$$

$$= \frac{3}{8}\left(\frac{4}{3}x^3 - \cancel{2}x^4 + \frac{1}{5}x^5\right)\Big|_0^2 =$$

OOPS

$$= \frac{3}{8}\left(\frac{16}{15}\right) = \frac{2}{5}$$

CENTER OF MASS $= \left(1, \frac{2}{5}\right)$

SOL'NS SUMMER '16

10. (20 points) We are given the probability density function for a random variable X as such:

$$f(x) = cx^2 e^{-\frac{1}{2}x}, \quad x \in [0, \infty)$$

and 0 otherwise.

(a) Find the value of c that makes $f(x)$ a probability density function (PDF)

TO ~~BE A~~ PDF, WE MUST FIND c SO THAT $\int_0^\infty f(x)dx = 1$.

$$\int x^2 e^{-x/2} dx = -2x^2 e^{-x/2} + 4\int x e^{-x/2} = -2x^2 e^{-x/2} + 4\left(-2xe^{-x/2} - 2\int xe^{-x/2}\right)$$

$$\boxed{\begin{array}{ll} u = x^2 & dv = e^{-x/2}dx \\ du = 2x\,dx & v = -2e^{-x/2} \end{array}} \qquad \boxed{\begin{array}{ll} u = x & dv = e^{-x/2} \\ du = dx & v = -2e^{-x/2} \end{array}}$$

$$= -2x^2 e^{-x/2} - 8xe^{-x/2} - 16e^{-x/2} + K.$$

SO $\int_0^\infty x^2 e^{-x/2}dx = \lim_{M \to \infty} \left(-e^{-M/2}(2M^2 + 8M + 16) + 16\right) = 0 + 16 = 16.$

SO $\boxed{c = \frac{1}{16}}$

(b) What is the $P(0 \le X \le 1)$? (You may leave your answer in terms of e)

$$\int_0^1 cx^2 e^{-x/2}dx$$

$$= c\left[(-2-8-16)e^{-1/2} + 16\right]$$

$$= \frac{1}{16}\left(16 - \frac{26}{\sqrt{e}}\right) = 1 - \frac{26}{16\sqrt{e}}$$

($16\sqrt{e}$ IS JUST BIGGER THAN 26 SO THIS IS OK)

SOL'NS SUMMER '16

11. (20 points (bonus)) Prove that, if $n \geq 1$,

$$\int \sin^n(x)dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$$

INTEGRATING BY PARTS,

$$\int \sin^n(x)dx = -\cos x \sin^{(n-1)} x + (n-1)\int \cos^2 x \sin^{(n-2)} x \, dx$$

WHERE $u = \sin^{n-1} x$ $\quad dv = \sin x \, dx$

$du = (n-1)\sin^{(n-2)} x \, dx$ $\quad v = -\cos x$

NOW USE $\cos^2 x = 1 - \sin^2 x$:

$$\int \sin^n(x)dx = -\cos x \sin^{(n-1)} x + (n-1)\int (1-\sin^2 x)\sin^{(n-2)} x \, dx$$

$$\int \sin^n(x)dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$$

ADD $(n-1)\int \sin^n x \, dx$ TO BOTH SIDES:

$$n\int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx$$

DIVIDE BY n TO GET RESULT:

$$\int \sin^n x \, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$$

SOL'NS SUMMER '16

12. (20 points (bonus)) Prove that, if $n \geq 1$ is an integer,

$$\int_0^\infty x^n e^{-x} dx = n!$$

(saying "by definition of the gamma function" does not count as proof)

INTEGRATE $\int x^n e^{-x} dx$ BY PARTS WITH $u = x^n$, $du = nx^{n-1}$, $dv = e^{-x}dx$, $v = -e^{-x}$

$$\int x^n e^{-x} dx = -e^{-x}x^n + n\int x^{n-1} e^{-x} dx$$

USE THIS RECURSIVELY TO SEE THAT

$$\int x^n e^{-x} dx = -e^{-x}x^n - e^{-x}(nx^{n-1}) - e^{-x}(n(n-1)x^{n-2}) - e^{-x} \cdot n(n-1)(n-2)x^{n-3} - \cdots - n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \cdot e^{-x}$$

THUS

$$\int_0^\infty x^n e^{-x} dx = \lim_{m \to \infty} \int_0^m x^n e^{-x} dx$$

$$= \lim_{m\to\infty}\left[-e^{-m}\left(x^n + nx^{n-1} + n(n-1)x^{n-2} + \cdots + n!\right) + e^0\left(0 + n\cdot 0 + \cdots + 0 + n!\right)\right]$$

$$= 0 + n! = n!$$