

Name: Sautons

Id: SPR 2010

20 pts

1. (a) Express the following limit of Riemann sums as a definite integral.  
(do not compute the integral)

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{x_i^2 + e^{x_i}} \Delta x, \quad \text{where } x_i = 2 + i\Delta x, \quad \Delta x = \frac{1}{n}$$

WITH  $\Delta x = \frac{1}{n}$  AND  $x_i = 2 + \frac{i}{n}$ , WE GET  $2 \leq x \leq 3$   
(TAKING  $i=0$  TO GET 2,  $i=n$  TO GET 3)

SO THE INTEGRAL IS

$$\int_2^3 \sqrt{x^2 + e^x} dx$$

(OTHER VARIATIONS LIKE  $\int_0^1 \sqrt{(2+x)^2 + e^{2+x}} dx$  ARE  
CORRECT, BUT UNEXPECTED)

20 pts

- (b) Express the following integral as a limit of Riemann sums.  
(do not compute the integral)

$$\int_1^3 \sqrt{1+x^3} dx$$

SINCE  $b-a = 3-1 = 2$ , IT IS REASONABLE

TO USE  $\Delta x = \frac{2}{n}$  AND SO  $x_i = 1 + \frac{2}{n}$ ,  $f(x) = \sqrt{1+x^3}$

THEN WE HAVE

$$\int_1^3 \sqrt{1+x^3} dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \cdot f(x_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \sqrt{1 + \left(1 + \frac{2}{n}\right)^3}$$

(OTHER VARIATIONS ARE POSSIBLE)

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30 pts 2. Define a function  $f(x) = \int_0^{x^2} \sqrt{t + \sqrt{t}} dt$ .

Find the value of  $f'(1)$ .

WRITE  $g(u) = \int_0^u \sqrt{t + \sqrt{t}} dt$ .

THEN BY THE FUNDAMENTAL THEOREM OF CALCULUS,

$$g'(u) = \sqrt{u + \sqrt{u}}$$

$$\text{SO, } f(x) = g(x^2) = \int_0^{x^2} \sqrt{t + \sqrt{t}} dt$$

AND BY THE CHAIN RULE,

$$f'(x) = g'(x^2) \cdot 2x = \left( \sqrt{x^2 + \sqrt{x^2}} \right) \cdot 2x = 2x \sqrt{x^2 + |x|}$$

THUS

$$f'(1) = 2 \cdot 1 \cdot \sqrt{1+1} = \boxed{2\sqrt{2}}$$

3. Determine whether each integral is convergent or divergent and evaluate those that are convergent (if any).

20 pts

(a)  $\int_{-\infty}^0 \frac{1}{(-1-x)^{1/3}} dx = \int_{-\infty}^0 \text{[scribble]} = \int_{-\infty}^0 -(1+x)^{-1/3} dx$

NOTE THAT THE INTEGRAND IS NOT DEFINED AT  $x=-1$ . WE NEED TO SPLIT THIS INTO THREE IMPROPER INTEGRALS:

ANY NUMBER  $\rightarrow$   $\leftarrow$   $< -1$  WORKS

$$\int_{-\infty}^{-9} -(1+x)^{-1/3} dx + \int_{-9}^{-1} -(1+x)^{-1/3} dx + \int_{-1}^0 -(1+x)^{-1/3} dx$$

$$= \lim_{m \rightarrow -\infty} \int_m^{-9} -(1+x)^{-1/3} dx + \lim_{t \rightarrow -1^-} \int_{-9}^t -(1+x)^{-1/3} dx + \lim_{r \rightarrow -1^+} \int_r^0 -(1+x)^{-1/3} dx$$

$$= \lim_{m \rightarrow -\infty} \left( -\frac{3}{2} (1+x)^{2/3} \right) \Big|_m^{-9} + \lim_{t \rightarrow -1^-} \left( -\frac{3}{2} (-8)^{2/3} + \frac{3}{2} t^{2/3} \right) + \lim_{r \rightarrow -1^+} \left( +\frac{3}{2} r^{2/3} \right)$$

20 pts

(b)  $\int_0^1 \frac{\ln x}{x} dx$

$$= \text{[scribble]} \left( -\frac{3}{2} \cdot 4 + \lim_{m \rightarrow -\infty} \frac{3}{2} (1+m)^{2/3} \right) + \frac{3}{2} \cdot 4 = \infty$$

INTEGRAL **DIVERGES!**

NOTE THAT  $\int \frac{\ln x}{x} dx = \int u du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$

$u = \ln x$   
 $du = \frac{dx}{x}$

BUT  $\frac{\ln x}{x}$  IS NOT DEFINED AT  $x=0$ , SO

$$\int_0^1 \frac{\ln x}{x} dx = \lim_{t \rightarrow 0^+} \int_t^1 \frac{\ln x}{x} dx = 0 - \lim_{t \rightarrow 0^+} \frac{(\ln t)^2}{2} = -\infty$$

SO INTEGRAL **DIVERGES**

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30 pts

4. Find the area of the region bounded by the two curves

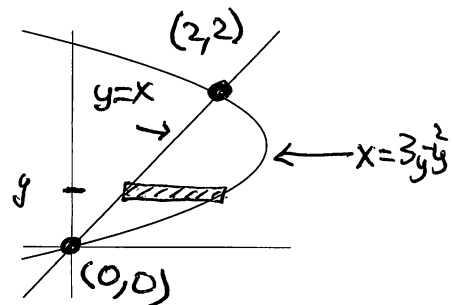
$$x = 3y - y^2 \quad \text{and} \quad y = x.$$

(a) Write an integral which represents this area.

THE CURVES CROSS IF  $3y - y^2 = y$

$$\text{ie } y^2 - 2y = 0$$

$$y(y-2) = 0, \quad \text{so } \begin{matrix} y = 0 & \text{or} & y = 2 \\ x = 0 & & x = 2. \end{matrix}$$



EASIEST TO INTEGRATE WRT  $y$

A TYPICAL RECTANGLE IS  $(3y - y^2) - y$  BY  $dy$

SO THE AREA IS GIVEN BY

$$\int_0^2 (2y - y^2) dy$$

(b) Evaluate the integral in (a).

$$\int_0^2 (2y - y^2) dy = y^2 - \frac{y^3}{3} \Big|_0^2$$

$$= \left(4 - \frac{8}{3}\right) - 0$$

$$= \frac{12}{3} - \frac{8}{3} = \boxed{\frac{4}{3}}$$

30 pts

5. Compute the following integral. If the integral diverges, write "divergent".

$$\int_{-1}^0 \frac{x}{(x-1)(x+2)} dx.$$

NOTE THAT  $\frac{x}{(x-1)(x-2)}$  IS DEFINED FOR  $x < 1$ , SO NOT IMPROPER.

USING PARTIAL FRACTIONS,

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

OR  $x = A(x-2) + B(x-1)$

IF  $x=1$ , THEN  $1 = -A + 0$ , so  $A = -1$

IF  $x=2$ , THEN  $2 = 0 + B$ , so  $B = 2$

OR

$$x = Ax - 2A + Bx - B$$

SO

$$A + B = 1$$

$$-2A - B = 0$$

ADDING GIVES

$$-A = 1, \text{ so } B = 2.$$

THUS

$$\int_{-1}^0 \frac{x}{(x-1)(x-2)} dx = \int_{-1}^0 \frac{-1}{x-1} + \frac{2}{x-2} dx$$

$$= -\ln|x-1| + 2\ln|x-2| \Big|_{-1}^0$$

$$= (-\ln(1) + 2\ln 2) - (-\ln(2) + 2\ln(3))$$

$$= 3\ln 2 - 2\ln 3$$

$$= \ln 8 - \ln 9 = \ln\left(\frac{8}{9}\right)$$

ANY OF THESE ARE OK.

30 pts 6. Find the volume of the solid obtained by rotating the region between the two curves

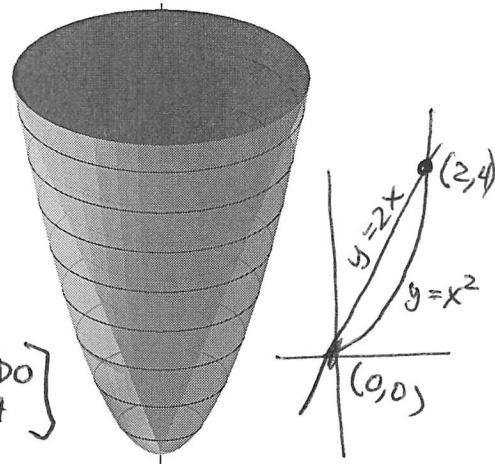
$$y = 2x \quad \text{and} \quad y = x^2$$

about the  $y$ -axis.

(a) Write an integral which represents the volume.

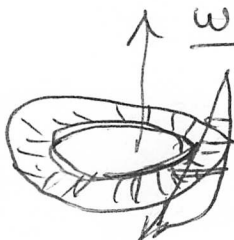
THE CURVES CROSS WHEN  $2x = x^2$ , THAT IS  
FOR  $x(x-2) = 0$ , AT  $x=0, x=2$   
 $y=0, y=4$ .

WE CAN INTEGRATE BY WASHERS ( $dy$ )  
OR CYLINDERS ( $dx$ ) [CHOOSE ONE, I'LL DO BOTH]

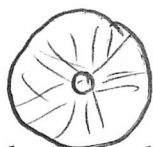


WASHERS ( $dy$ )

OUTER CURVE IS  $x = \sqrt{y}$   
INNER IS  $x = y/2$



A SLICE IS A RING/WASHER WITH AREA



$$\pi(\sqrt{y})^2 - \pi(y/2)^2$$

OVER                      INNER

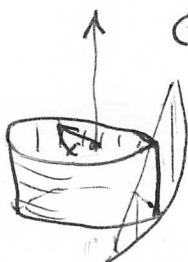
INTEGRAL IS

$$\int_0^4 \pi(y - y^2/4) dy$$

(b) Evaluate the integral in (a).

CYLINDERS ( $dx$ )

SLICES ARE VERTICAL,  
TOP IS  $y = 2x$   
BOTTOM IS  $y = x^2$  } HEIGHT IS  $2x - x^2$



CIRCUMFERENCE IS  $2\pi x$

AREA IS  $2\pi x(2x - x^2)$

INTEGRAL IS

$$\int_0^2 2\pi(2x^2 - x^3) dx$$

(b) EVALUATE

FOR WASHERS,

$$\pi \int_0^4 (y - y^2/4) dy = \pi \left( \frac{y^2}{2} - \frac{y^3}{12} \right) \Big|_0^4 = \pi \left( 8 - \frac{64}{12} \right) = \boxed{\frac{8\pi}{3}}$$

FOR CYLINDERS

$$2\pi \int_0^2 (2x^2 - x^3) dx = 2\pi \left( \frac{2}{3}x^3 - \frac{1}{4}x^4 \right) \Big|_0^2 = 2\pi \left( \frac{16}{3} - \frac{16}{4} \right) = \boxed{\frac{8\pi}{3}}$$

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20 pts 7. (a) Compute the definite integral

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

LET  $u = \sqrt{x} = x^{1/2}$   
 $du = \frac{1}{2} x^{-1/2} dx = \frac{dx}{2\sqrt{x}}$

$x=1 \Rightarrow u=1$   
 $x=4 \Rightarrow u=2$

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int_1^2 2e^u du = 2e^u \Big|_1^2 = \boxed{2(e^2 - e)}$$

20 pts

(b) Compute the indefinite integral  $\int (\sec 2t)(\tan 2t) dt$

SINCE  $\frac{d}{dx} \tan(x) = \sec(x) \tan(x)$ ,

LET  $u = 2t$   
 $du = 2dt$

$$\begin{aligned} & \int \sec(2t) \tan(2t) dt \\ &= \frac{1}{2} \int \sec u \tan u du = \frac{1}{2} \tan u + C \\ &= \boxed{\frac{1}{2} \tan(2t) + C} \end{aligned}$$

20 pts

8. (a) Compute the following indefinite integral  $\int p^6 \ln p \, dp$ INTEGRATE BY PARTS. ( $\int u \, dv = uv - \int v \, du$ )

$$u = \ln p \quad dv = p^6 \, dp$$

$$du = \frac{1}{p} \, dp \quad v = \frac{1}{7} p^7$$

$$\int p^6 \ln p \, dp = \frac{1}{7} p^7 \ln p - \frac{1}{7} \int p^6 \, dp$$

$$= \boxed{\frac{1}{7} p^7 \ln p - \frac{1}{49} p^7 + C}$$

20 pts

(b) Compute the following definite integral  $\int_0^{\pi} t \sin 3t \, dt$ .

AGAIN, BY PARTS.

$$u = t \quad dv = \sin 3t \, dt$$

$$du = dt \quad dv = -\frac{1}{3} \cos 3t$$

$$\int_0^{\pi} t \sin 3t \, dt = -\frac{1}{3} t \cos(3t) \Big|_0^{\pi} - \int_0^{\pi} \left(-\frac{1}{3} \cos 3t\right) dt$$

$$= -\frac{1}{3} t \cos(3t) + \frac{1}{9} \sin 3t \Big|_0^{\pi}$$

$$= \left(-\frac{1}{3} \pi \cos 3\pi + \frac{1}{9} \sin 3\pi\right) - \left(-\frac{1}{3} \cdot 0 \cdot \cos 0 + \frac{1}{9} \sin 0\right)$$

$$= -\frac{1}{3} \pi (-1) + \frac{1}{9} (0) - 0$$

$$= \boxed{\frac{\pi}{3}}$$



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9. (a) Write an equation in Cartesian ( $x$ - $y$ ) coordinates for the curve with polar equation  $r = 2 \sin \theta$ . Your answer should not contain trigonometric functions.

RECALL THAT  $x = r \cos \theta$      $r^2 = x^2 + y^2$   
 $y = r \sin \theta$      $\tan \theta = y/x$

MULTIPLY BOTH SIDES BY  $r$  TO GET

$$r^2 = 2r \sin \theta$$

SO

$$x^2 + y^2 = 2y$$

THIS IS OK, BUT WE CAN GET THE FAMILIAR FORM OF A CIRCLE WITH SOME REARRANGING.

$$x^2 + y^2 - 2y = 0$$

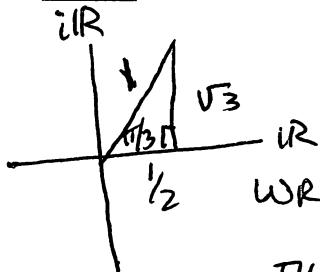
$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y-1)^2 = 1$$

THIS IS A CIRCLE OF RADIUS 1 AND CENTER  $(0, 1)$ .

20 pts

- (b) Find the  $(a + ib)$ -form of the complex number



$$\left[ \frac{1 + i\sqrt{3}}{2} \right]^{20}$$

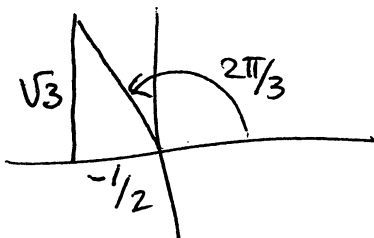
WRITE  $\frac{1}{2} + \frac{i\sqrt{3}}{2}$

IN POLAR FORM, AS  $e^{i\pi/3}$

THEN

$$\left[ \frac{1}{2} + \frac{i\sqrt{3}}{2} \right]^{20} = \left[ e^{i\pi/3} \right]^{20} = e^{\frac{20\pi i}{3}} = e^{(6\pi + \frac{2\pi}{3})i} = e^{\frac{2\pi i}{3}}$$

CONVERTING BACK TO  $a + ib$ , WE GET

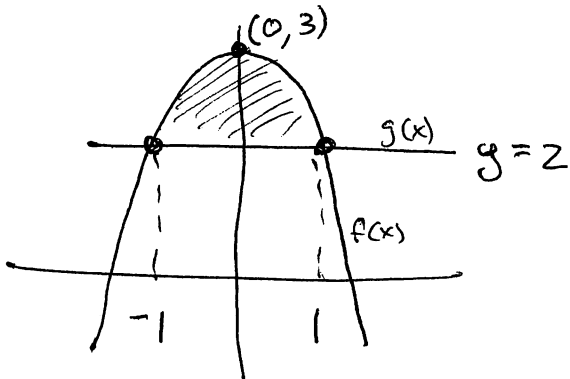


$$e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

( $6\pi$  IS 3 FULL TURNS)

- 30 pts 10. Find the center of mass of a flat plate with uniform density that occupies the region bounded by the two curves  $y = 2$  and  $y = 3 - x^2$ .

[ THIS TOPIC IS NOT ON THE FALL 2016 EXAM ]



NEED TO FIND THE AVERAGE VALUES  $\bar{x}$  AND  $\bar{y}$

THE TWO CURVES CROSS AT  $(-1, 2)$  AND  $(1, 2)$

TOTAL AREA IS

$$\int_{-1}^1 (3 - x^2) - 2 \, dx = \int_{-1}^1 (1 - x^2) \, dx = x - \frac{x^3}{3} \Big|_{-1}^1 = \frac{4}{3} = A.$$

$$\bar{x} = \frac{1}{A} \int_{-1}^1 x(f(x) - g(x)) \, dx = \frac{3}{4} \int_{-1}^1 x - x^3 \, dx = \frac{3}{4} \left( x^2 - \frac{x^4}{4} \right) \Big|_{-1}^1 = 0$$

$$\begin{aligned} \bar{y} &= \frac{1}{A} \int_{-1}^1 \frac{1}{2} ([f(x)]^2 - [g(x)]^2) \, dx = \frac{3}{8} \int_{-1}^1 ((3 - x^2)^2 - 2^2) \, dx \\ &= \frac{3}{8} \int_{-1}^1 (9 - 6x^2 + x^4 - 4) \, dx \\ &= \frac{3}{8} \left( 5x - 2x^3 + \frac{x^5}{5} \right) \Big|_{-1}^1 \\ &= \frac{3}{8} \cdot \frac{32}{5} = \frac{96}{40} = \frac{12}{5} \end{aligned}$$

CENTER OF MASS IS  $(0, \frac{12}{5})$