MATH 126 Solutions to Midterm 2 (Venkman)

30 pts 1. Evaluate each of the integrals below. If the integral does not converge, write **Diverges**.

(a) $\displaystyle\int_0^1 \frac{5x+1}{1+x^2}\,dx$

Solution:

$$\int_0^1 \frac{5x+1}{1+x^2}\,dx = \int_0^1 \frac{5x\,dx}{1+x^2} + \int_0^1 \frac{dx}{1+x^2}$$
$$= 5\int_1^2 \frac{du/2}{u} + \arctan(x)\Big|_0^1 = \tfrac{5}{2}\ln|u|\Big|_1^2 + \left(\frac{\pi}{4} - 0\right) = \tfrac{5}{2}\ln|2| + \frac{\pi}{4},$$

where we made the substitution $u = 1 + x^2$, $du = 2x\,dx$ in the first integral, and adjusted the bounds accordingly.

(b) $\displaystyle\int \sin^5(x)\cos^3(x)\,dx$

Solution: We use the trigonometric identity $\cos^2 x = 1 - \sin^2 x$ so that we can make the substitution $u = \sin x$, $du = \cos x\,dx$.

$$\int \sin^5(x)\cos^3(x)\,dx = \int \sin^5(x)(1-\sin^2(x))\cos(x)\,dx$$
$$= \int u^5(1-u^2)\,du = \int u^5 - u^7\,du = \frac{\sin^6(x)}{6} - \frac{\sin^8(x)}{8} + C.$$

(c) $\displaystyle\int_1^\infty \frac{dx}{x^3}$

Solution:

$$\int_1^\infty \frac{dx}{x^3} = \lim_{M\to\infty}\int_1^M \frac{dx}{x^3} = \lim_{M\to\infty} \frac{-1}{4x^4}\Bigg|_1^M = \lim_{M\to\infty}\left(\frac{-1}{4M^4} + \frac{1}{4}\right) = \left(0 + \frac{1}{4}\right) = \frac{1}{4}.$$

30 pts 2. Recall that $\int_1^3 \frac{dx}{x} = \ln 3$.

(a) Use Simpson's rule with $n = 4$ (that is, 5 points) to approximate $\ln 3$.

Solution: Since we are using 4 intervals, each is $\frac{1}{2}$ wide, and $f(x) = \frac{1}{x}$.

$$\ln 3 \approx \frac{1}{2} \cdot \frac{1}{3} \left(f(1) + 4f(3/2) + 2f(2) + 4f(5/2) + f(3)\right)$$
$$= \frac{1}{6}\left(1 + \frac{4 \cdot 2}{3} + \frac{2}{2} + \frac{4 \cdot 2}{5} + \frac{1}{3}\right) = \frac{1}{6}\left(1 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{1}{3}\right) = \frac{33}{30} = \frac{11}{10} = 1.1$$

(b) Estimate the error in your approximation above. The maximum error in Simpson's rule to approximate $\int_a^b f(x)\,dx$ is $\frac{M(b-a)^5}{180n^4}$, where M is the maximum of $|f^{(4)}(x)|$ for $a \leq x \leq b$.

Solution: We know $n = 4$ and $b - a = 2$, but we need to figure out M, which requires us to take some derivatives.
$f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$, and $f^{(4)}(x) = 24x^{-5}$.
Since $1/x^5$ is decreasing for $1 < x < 3$, the largest value will occur at $x = 1$, which is 24. Thus the error is at most

$$\frac{24 \cdot 2^5}{180 \cdot 4^4} = \frac{(2 \cdot 3 \cdot 4) \cdot 2^5}{(4 \cdot 3 \cdot 15) \cdot 2^6} = \frac{1}{60}.$$

(In fact, the approximation $\ln(3) \approx 1.1$ is about 0.0013877 too large, while $1/60 \approx 0.16667$. But you don't need to know that.)

(c) What n will ensure that Simpson's rule gives an answer correct to within ± 0.001?*

Solution: From the previous part, we know we can use $M = 24$. This means we need to find an even value of n (since Simpson's requires n to be even) so that

$$\frac{24 \cdot 2^5}{180 \cdot n^4} < 0.001.$$

This means we want to have

$$n^4 > \frac{24 \cdot 2^5}{180 \cdot 0.001} = \frac{24 \cdot 2^5 \cdot 1000}{12 \cdot 5 \cdot 3} = \frac{12800}{3} \approx 4266.$$

We can see from the list given that we need $n > 9$ ($n = 8$ is just a bit too small), so we must take $n = 10$. (since n has to be an even whole number).

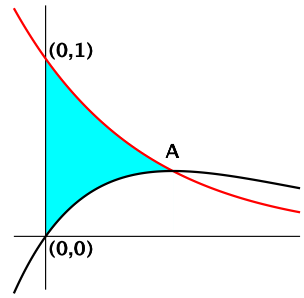
* $4^4 = 256, 5^4 = 625, 6^4 = 1296, 7^4 = 2401, 8^4 = 4096, 9^4 = 6561, 10^4 = 10000, 11^4 = 14641, 12^4 = 20736, 13^4 = 28561, 14^4 = 38416, 15^4 = 50625, 16^4 = 65535, 17^4 = 83521, 18^4 = 104976, 19^4 = 130321, 20^4 = 160000.$

30 pts

3. Consider the finite region in the first quadrant bounded by the curves

$$y = e^{-2x} \quad \text{and} \quad y = xe^{-2x},$$

corresponding to the shaded region in the figure at right.



(a) On the figure, label both curves and the three points at the "corners" of the region.

Solution: The curve that passes through the origin (in black) is $y = xe^{-2x}$, and the red curve is $y = e^{-2x}$. They cross at the point marked **A**. Since $e^{-2x} = xe^{-2x}$ exactly when $(x-1)e^{-2x} = 0$, the only solution is $x = 1$. Thus, $\mathbf{A} = (1, e^{-2})$.

(b) Write an integral which represents the area of the region.

Solution:

$$\int_0^1 e^{-2x} - xe^{-2x}\, dx = \int_0^1 (1-x)e^{-2x}\, dx$$

(c) Calculate the area of the region.

Solution: Integrate by parts, letting $u = (1-x)$ and $dv = e^{-2x}\, dx$, so that $du = -dx$ and $v = -\frac{1}{2}e^{-2x}$. Thus

$$\int_0^1 (1-x)e^{-2x}\, dx = -\frac{1-x}{2e^{2x}}\Bigg|_0^1 - \int_0^1 \frac{1}{2}e^{-2x}\, dx$$

$$= -\frac{1-x}{2e^{2x}} + \frac{1}{4e^{2x}}\Bigg|_0^1 = \left(0 + \frac{1}{4e^2}\right) - \left(-\frac{1}{2} + \frac{1}{4}\right) = \frac{1}{4e^2} + \frac{1}{4}$$

30 pts

4. Zhulong has a candle store, and he sells tapered candles with a star-shaped cross-section, like the one shown at right.
He is setting up a process to manufacture these candles (he used to make them by hand), and so he needs a careful calculation of the volume of each one. The candles have sides that are tapered so that the diameter at height h is $4 - h/4$. The area of a regular five-pointed star inscribed in a circle of radius r is given by the formula

$$A(r) = \frac{10\tan\left(\frac{\pi}{10}\right)}{3 - \tan^2\left(\frac{\pi}{10}\right)}\, r^2 \;\approx\; \frac{9}{8}r^2.$$

What is the volume of a candle that is 4 inches tall?
(The approximation $\frac{9}{8}$ is within $.001$ of the true value, so use that. You don't have to simplify the fractions.)

Solution: Remember that if we know how to find the area of all slices of an object, we can compute its volume by integrating those areas. In this case, horizontal slices are star-shaped, and the area of a slice with radius r is very close to $\frac{9}{8}r^2$. This means that

$$\text{volume of candle of height 4} \;\approx\; \int_0^4 \frac{9}{8}r^2\, dh.$$

The thing to notice is that the integral is with respect to h, not r, so we have to write r in terms of h. (Many people tried to integrate with respect to r, which makes the slices very complicated – some would be thin cylinders, some would be a collection of 5 curved rectangles or triangles of various widths an heights – not easy. Nobody who tried this noticed that.)

So, since the diameter at height h is given by $4 - h/4$, the radius is half of that; $r = 2 - h/8$. This means we have

$$\text{Vol} = \int_0^4 \frac{9}{8}\left(2 - \frac{h}{8}\right)^2 dh = \frac{9}{8}\int_2^{3/2} (-8\,u^2)\, du \quad \text{(substituting } u = 2 - \frac{h}{8}\text{, so } -8du = dh\text{)}$$

$$= -9\frac{u^3}{3}\Bigg|_2^{3/2} = -3\left(\frac{3^3}{2^3} - 8\right) = 3 \cdot \frac{37}{8} = \frac{111}{8} = 13.875 \text{ cubic inches}$$

As noted in the problem, you don't need to simplify the horrible fractions. We have calculators for that.

30 pts 5. Compute the integral: $\displaystyle\int_2^3 \frac{4x^3 - 5x^2 - 2x + 1}{x^2(1+x)(1-x)}\,dx$.

Remember that an *exact* answer is wanted; do not approximate logarithms, exponentials, π, or square roots.

Solution: Using partial fractions decomposition, we have

$$\frac{4x^3 - 5x^2 - 2x + 1}{x^2(1+x)(1-x)} = \frac{Ax+B}{x^2} + \frac{C}{1+x} + \frac{D}{1-x}.$$

Don't forget that since there is a quadratic term (x^2) in one of the factors, its numerator needs to be of the form $Ax + B$. We need to solve

$$4x^3 - 5x^2 - 2x + 1 = (Ax+B)(1+x)(1-x) + C(x^2)(1-x) + D(x^2)(1+x)$$

for A, B, C, and D. Since it must hold for all x, we can choose values of x to make life easier.

When $x = 0$, we get $1 = B$ right away.

When $x = 1$, we have $4 - 5 - 2 + 1 = 2D$, so $2D = -2$, and thus $D = -1$.

When $x = -1$, we get $-4 - 5 + 2 + 1 = -2C$, so $-2C = -6$ and $C = 3$.

Finally, we have to figure out A. Here we choose any other value of x that we haven't used yet. Taking $x = 2$ gives us $32 - 20 - 4 = (2A+B)(-3) + C(-4) - D(12)$, and remembering that $B = 1$, $C = -3$, and $D = -1$ yields $9 = -6A - 3$. So we know $A = -2$.

(If you prefer to multiply everything out and solve the resulting equations, you can. You should get the same answer. I don't want to do that, because it is long and I will blow it.)

This means we have

$$\begin{aligned}
\int_2^3 \frac{4x^3 - 5x^2 - 2x + 1}{x^2(1+x)(1-x)}\,dx &= \int_2^3 \frac{1-2x}{x^2} - \frac{1}{1-x} + \frac{3}{1+x}\,dx = \int_2^3 \frac{1}{x^2} - \frac{2}{x} + \frac{1}{x-1} + \frac{3}{1+x}\,dx \\
&= -\frac{1}{x} - 2\ln|x| + \ln|x-1| - 3\ln|1+x|\Big|_2^3 \\
&= \left(-\frac{1}{3} - 2\ln 3 + \ln 2 - 3\ln 4\right) - \left(-\frac{1}{2} - 2\ln 2 + \ln 1 - 3\ln 3\right) \\
&= \frac{1}{6} + \ln 3 + 4\ln 2 - 3\ln 4 = \frac{1}{6} + \ln 3 - 3\ln 2,
\end{aligned}$$

where we remembered that $\ln 1 = 0$ and $\ln 4 = 2\ln 2$ in order to simplify.

30 pts 6. Compute the integral: $\displaystyle\int_0^{\frac{\sqrt{2}}{2}} \frac{6x^2\ dx}{\sqrt{1-x^2}}$.

Remember that an *exact* answer is wanted; do not approximate logarithms, exponentials, π, or square roots.

Solution: This is an integral that is best done via a trigonometric substitution, which enables us to get rid of the $\sqrt{1-x^2}$ term.

Let $x = \sin\theta$ and so $dx = \cos\theta\ d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$. Since this is a definite integral, we notice that when $x = 0$ we have $\theta = 0$, and when $x = \frac{\sqrt{2}}{2}$, $\theta = \frac{\pi}{4}$.

Thus

$$\begin{aligned}
\int_0^{\frac{\sqrt{2}}{2}} \frac{6x^2\ dx}{\sqrt{1-x^2}} &= \int_0^{\pi/4} \frac{6\sin^2\theta\ \cos\theta\, d\theta}{\cos\theta} \\
&= 6\int_0^{\pi/4} \sin^2\theta\ d\theta = 6\int_0^{\pi/4} \tfrac{1}{2}(1-\cos(2\theta))d\theta \\
&= 3\big(\theta - \tfrac{1}{2}\sin(2\theta)\big)\Big|_0^{\pi/4} = 3\Big[\Big(\frac{\pi}{4} - \tfrac{1}{2}\sin(\pi/2)\Big) - \big(0 - \tfrac{1}{2}\sin(0)\big)\Big] = \frac{3\pi}{4} - \frac{3}{2}.
\end{aligned}$$