SOLUTIONS.

Part 2: These will be graded **only** if you have passed part 1. **Name:** ____________

13. Find an antiderivative (that is, a function whose derivative is the given function) for each of the following functions:

3 points (a) $f(x) = 3x^3 - 6x^2 + x + e^2$

$$\boxed{\frac{3}{4}x^4 - 2x^3 + \frac{1}{2}x^2 + e^2x + C}$$

(REMEMBER e^2 IS A NUMBER BETWEEN 7 & 8)

(a) ____________

3 points (b) $g(x) = \sqrt{2+x}$

$= (2+x)^{1/2}$

ANTIDERIV IS $\boxed{\frac{2}{3}(2+x)^{3/2} + C}$

(b) ____________

3 points (c) $h(x) = \frac{x}{5} - \frac{5}{x} = \frac{1}{5}x - 5\cdot\frac{1}{x}$

ANTIDERIV IS $\boxed{\frac{1}{10}x^2 - 5\ln|x| + C}$

(c) ____________

3 points (d) $k(x) = \frac{2}{\sqrt{1-x^2}} =$

$$\boxed{2\arcsin(x) + C}$$

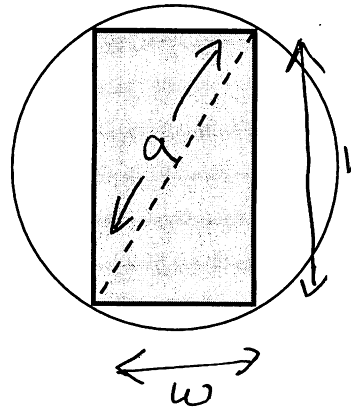
(d) ____________

Part 2: These will be graded **only** if you have passed part 1. **Name:** ______________

12 points | 14. The strength of a rectangular beam of a given length is proportional to the width times the square of the height of a cross-section, that is,

$$S = w \cdot h^2$$

where S is the strength (in some appropriate units), w is the width, and h is the height.
Find the dimensions of the strongest beam that can be cut out of a log which has a circular cross-section with a 9" diameter (that is, the cross-section of the beam is a rectangle with a diagonal of 9").



WANT TO MAXIMIZE $w \cdot h^2$

BUT WITH $w^2 + h^2 = 81$ (AND $0 < w < 9$)

SO $h^2 = 81 - w^2$.

STRENGTH IS $S(w) = w(81 - w^2) = 81w - w^3$

CRIT. PT. WHEN $S'(w) = 0$, IE

$$81 - 3w^2 = 0$$

SO $81 = 3w^2$

$27 = w^2$, $w = \pm 3\sqrt{3}$

(BUT ONLY $+3\sqrt{3}$ IN RANGE)

$w = \sqrt{3}$ IS MAX, SINCE

$S''(w) = -6w$, $S''(3\sqrt{3}) < 0$

SO CRIT. PT IS REL MAX.

WANT DIMENSIONS,

SO $h = \sqrt{81 - (3\sqrt{3})^2} = \sqrt{54} = 3\sqrt{6}$

BEAM SHOULD BE WIDTH $3\sqrt{3}$ BY HEIGHT $3\sqrt{6}$.

(about 5.2" wide by 7.3" tall)

Part 2: These will be graded **only** if you have passed part 1. **Name:** ________________

16. Consider the function $f(x) = 3x^{\frac{2}{3}} - 2x$.

4 points (a) Find the x-values of all critical points of $f(x)$

$f'(x) = 2x^{-1/3} - 2$.

$f'(x) = 0 \Leftrightarrow x^{-1/3} = 1$, so $x=1$ IS A CRIT POINT

BUT ALSO,

$f'(x)$ DNE AT $x=0$.

CRITS ARE $x=0$, $x=1$.

4 points (b) State the largest interval on which $f(x)$ is increasing.

SINCE THE CRITICAL POINTS ARE 0 & 1,
MUST CHECK

$x<0$: $f'(x) < 0$ DECREASING. (CUBE ROOT OF NEG IS NEG)

$0 < x < 1$: $f'(x) > 0$ INCREASING. (REMEMBER FOR $0<x<1$ $x^{-1/3} > 1$)

$1 < x$: $f'(x) < 0$, DECREASING.

INTERVAL OF INCREASE IS $(0, 1)$

4 points (c) Give the x-values at which the absolute maximimum and absolute minimum values of f occur when $-1 \le x \le 3$.
(You might find it helpful to know that $2^{\frac{2}{3}} \approx 1.59$, $3^{\frac{2}{3}} \approx 2.08$, and $4^{\frac{2}{3}} \approx 2.52$.)

MUST CHECK AT $x=-1$, $x=0$, $x=1$, $x=3$.

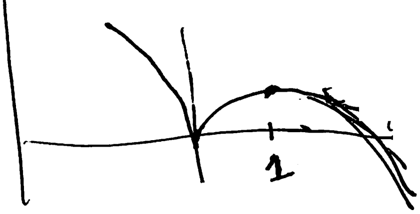
$f(-1) = 3+2 = 5$ ← MAX

$f(0) = 0$ ← MIN

$f(1) = 3-2 = 1$

$f(3) = 3 \cdot 3^{2/3} - 6 \approx 0.24$

GRAPH LOOKS LIKE



MAX AT $x=-1$, MIN AT $x=0$

Part 2: These will be graded **only** if you have passed part 1. **Name:** ______________

17. Jimi Chiu makes "designer" shoes[1] that he sells at \$150 a pair. He knows that the number of pairs of shoes he sells is a function of the price he charges; let's denote this by $N(p)$, where p is the price per pair. Market research tells him that $N'(150)$ is about -10; that is, if he raises the price by one dollar, he should expect to sell 10 fewer pairs. The amount of revenue $R(p)$ he makes at a given price will be given by $R(p) = p \cdot N(p)$.

7 points (a) If he typically sells 2000 pairs of shoes at \$150 each, what is $R'(150)$?

$R(P) = P \cdot N(P)$, so $R'(P) = N(P) + P N'(P)$.

~~WE WANT~~ SELLING 2000 PAIRS AT \$150 EACH MEANS

$N(150) = 2000$

$R'(150) = 2000 + (150)(-10) = 2000 - 1500$
$= 500$

5 points (b) Use your answer above to estimate his revenue if he raises the price to \$160 per pair (that is, estimate $R(160)$). Should he raise the price?

APPROXIMATING R BY THE TANGENT LINE GIVES

$R(160) \approx R(150) + R'(150) \cdot (10)$

$= (2000)(150) + (500)(10)$

$= 300\,000 + 5\,000 = \$305,000.$

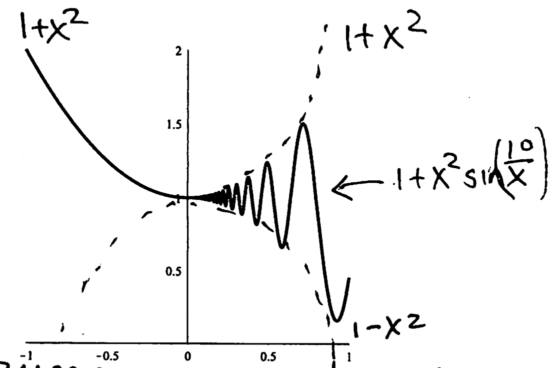
BY RAISING HIS PRICE \$10/PAIR, HE SHOULD MAKE \$5,000 MORE IN REVENUE, SO YES.

[1]No relation to Jimmy Choo shoes, unless you don't look very closely. Mr. Chiu is also fond of Rollexx watches.

18. Consider the function

$$Q(x) = \begin{cases} 1 + x^2 & x \le 0 \\ 1 + x^2 \sin(10/x) & \text{otherwise} \end{cases}$$

with the graph shown at right.



6 points (a) Show that $Q(x)$ is continuous at every value of x.

FOR $X<0$, $Q(X) = 1+X^2$ WHICH IS CONTINUOUS SINCE ITS A POLYNOMIAL
FOR $X>0$, $Q(x) = 1+x^2\sin\left(\frac{10}{x}\right)$, ALSO CONTINUOUS FOR ALL $X \neq 0$.
WE JUST NEED TO CHECK THAT

$Q(0) = \lim_{x\to 0} Q(x)$. NOTE $Q(0) = 1+0 = 1$

$\lim_{x\to 0^-} Q(x) = \lim_{x\to 0^-} (1+x^2) = 1$.

$\lim_{x\to 0^+} Q(x) = \lim_{x\to 0^+} 1+x^2\sin\left(\frac{10}{x}\right) = 1$ BY SQUEEZE THM.

IF $X \neq 0$,
$1-x^2 \le 1+x^2\sin\left(\frac{10}{x}\right) \le 1+x^2$
SO
$1 = \lim_{x\to 0}(1-x^2)$
$\le \lim_{x\to 0} 1+x^2\sin\left(\frac{10}{x}\right)$
$\le \lim_{x\to 0}(1+x^2) = 1$
SO $\lim_{x\to 0} 1+x^2\sin\frac{10}{x} = 1$.

SINCE $Q(0) = \lim_{x\to 0} Q(0) = 1$, Q IS CONTINUOUS

6 points (b) Is $Q(x)$ differentiable at $x = 0$? If so, what is $Q'(0)$? If not, why not?

MUST USE DEF. OF DERIVATIVE!

$$Q'(0) = \lim_{h\to 0} \frac{[Q(0+h)] - Q(0)}{h}$$

$$= \lim_{h\to 0} \frac{\left[1+h^2\sin\left(\frac{10}{h}\right)\right] - 1}{h} = \lim_{h\to 0} \frac{h^2\sin\left(\frac{10}{h}\right)}{h} = \lim_{h\to 0} h\sin\left(\frac{10}{h}\right)$$

BY SQUEEZE THM; FOR $h \neq 0$:

$$-h \le h\sin\left(\frac{10}{h}\right) \le h$$

SO

$$0 = \lim_{h\to 0}(-h) \le \lim_{h\to 0} h\sin\left(\frac{10}{h}\right) \le \lim_{h\to 0} h = 0.$$

SO

$$\boxed{Q'(0) = 0.}$$