MAT 125-Final Exam (part 2) -FALL 2017

NAME: SOLUTIONS.

TA NAME:

*Each numbered question is worth 20% of the exam.

1. Find and draw f if $f'(x) = 5 - 4x - x^2$ and $f(0) = 2$.

"FIND" MEANS WE NEED THE ANTIDERIVATIVE $f(x)$ WITH $f(0)=2$.

SO $f(x) = 5x - 2x^2 + \frac{1}{3}x^3 + C$, AND SINCE $f(0)=2$, $C=2$.

SO $\boxed{f(x) = 5x - 2x^2 + \frac{1}{3}x^3 + 2}$

$f'(x) = 5 - 4x - x^2 = (5+x)(1-x)$, SO CRIT PTS AT $x=-5$, $x=-1$

$f'(x) < 0$ IF $x < -5$ OR IF $x > +1$

$f'(x) > 0$ IF $-5 < x < +1$

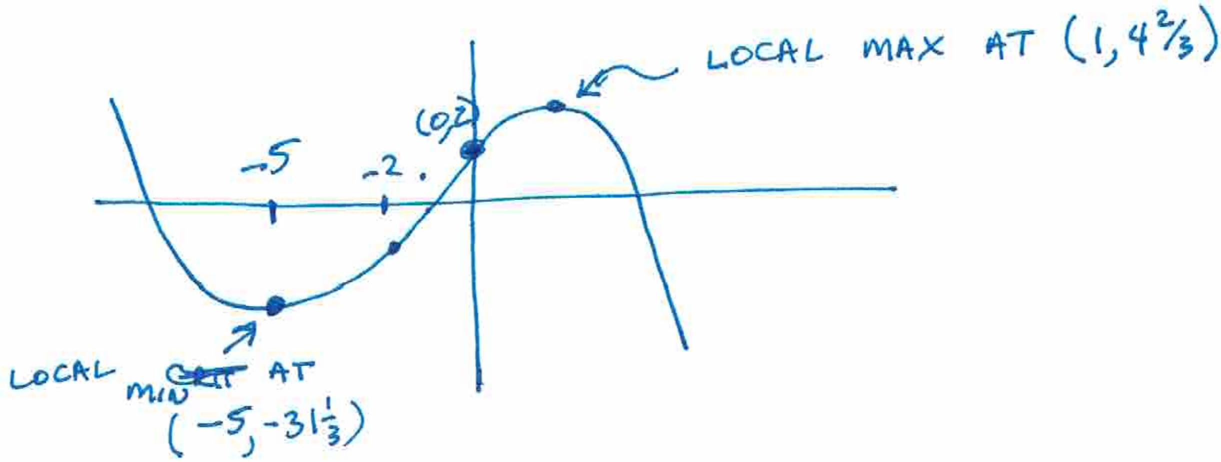
SO f IS DECREASING ON $(-\infty, -5) \cup (1, +\infty)$

INCREASING ON $(-5, 1)$

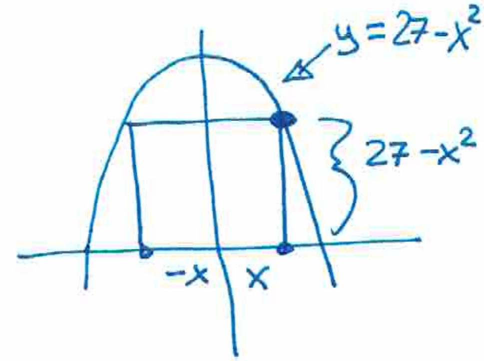
$f''(x) = -4 - 2x$, SO $x = -2$ IS AN INFLECTION POINT.

(f CONCAVE UP FOR $x < -2$, DOWN FOR $x > -2$)

GRAPH LOOKS LIKE



2) A rectangle has its base on the x axis and its upper two vertices on the the parabola $y = 27 - x^2$. What is the largest area the rectangle can have?

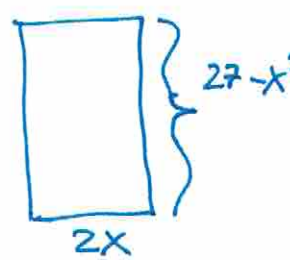


IT SHOULD BE "OBVIOUS" THAT THE RECTANGLE WILL BE SYMMETRIC WRT. THE y AXIS (IF NOT, BOTH UPPER VERTICES WILL NOT LIE ON THE PARABOLA)

SO, LET THE LOWER RIGHT CORNER BE $(x, 0)$ AND THE LOWER LEFT BE $(-x, 0)$. [SO THE WIDTH IS $2x$].

THEN SINCE THE UPPER RIGHT CORNER WILL BE ON THE PARABOLA, IT HAS COORDINATES

$(x, 27 - x^2)$. IN PARTICULAR, THE HEIGHT IS $27 - x^2$.



WE WANT TO MAXIMIZE

$A(x) = 2x(27 - x^2)$ WITH $0 < x < \sqrt{27}$

$A(x) = 54x - 2x^3$

FIND CRITICAL POINT(S):

$A'(x) = 54 - 6x^2$.

$A'(x) = 0 \iff 54 = 6x^2 \iff 9 = x^2 \iff x = \pm 3$.

SINCE WE WANT $0 < x < \sqrt{27}$, $x = +3$ IS ONLY RELEVANT CRIT.

$A''(x) = -12x$, SO $A''(3) < 0$

HENCE $x = 3$ IS A LOCAL MAX.

BEST DIMENSIONS ARE 6 BY 18

IF $x = 3$, WIDTH IS 6

HEIGHT IS $27 - 3^2 = 18$

$\boxed{\text{MAX AREA IS } 108}$

3) Determine the following limits or explain why they do not exist if $f(x) = \frac{e^x}{x}$

a) $\lim_{x \to 0} f(x)$

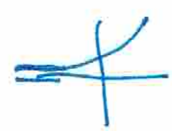
$\lim_{x \to 0} \frac{e^x}{x}$ DNE, SINCE $\lim_{x \to 0^+} \frac{e^x}{x} = +\infty$, BUT

$\lim_{x \to 0^-} \frac{e^x}{x} = -\infty$

($e^0 = 1$, SO WE DIVIDE A NUMBER NEAR +1 BY A NUMBER NEAR 0, EITHER + OR −)

b) $\lim_{x \to -\infty} f(x)$

$$\lim_{x \to -\infty} \frac{e^x}{x} = \left(\lim_{x \to -\infty} e^x\right)\left(\lim_{x \to -\infty} \frac{1}{x}\right) = 0 \cdot 0 = 0.$$

[e^x LOOKS LIKE  AND $\frac{1}{x}$ LOOKS LIKE 

BOTH HAVE 0 AS A HORIZ ASYMPTOTE ON THE LEFT]

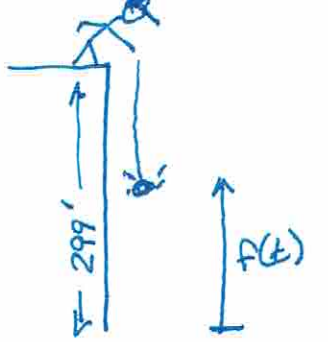
c) $\lim_{x \to \infty} \sin(f(x))$

$\lim_{x \to \infty} \sin\left(\frac{e^x}{x}\right) = \sin\left(\lim_{x \to \infty} \frac{e^x}{x}\right)$, SINCE $\sin x$ IS CONTINUOUS.

$\lim_{x \to \infty} \frac{e^x}{x}$ IS OF THE FORM $\frac{\infty}{\infty}$, SO WE CAN USE L'HÔPITAL

TO GET $\lim_{x \to \infty} \frac{e^x}{1} = +\infty$. BUT AS $x \to \infty$, $\lim_{x \to \infty} \sin x$ OSCILLATES,

SO $\lim_{x \to \infty} \sin\left(\frac{e^x}{x}\right)$ DNE.



4) A boy at the top of a cliff 299 ft.high throws a rock straight down, and it hits the ground 3.25 seconds later.With what speed does the boy throw the rock? The gravitational constant is -32 ft/s^2.

SINCE THE GRAV. CONSTANT IS -32 ft/s², THIS MEANS THE ACCELERATION IS CONSTANT -32.

IF $f(t)$ IS THE POSITION OF THE ROCK AT t SECONDS, $f'(t)$ WILL BE THE VELOCITY AT t, AND $f''(t)$ THE ACCEL.

SO

$$f''(t) = -32$$
$$f'(t) = -32t + C$$
$$f(t) = -16t^2 + Ct + K.$$

IF WE LET $t=0$ BE WHEN THE ROCK IS THROWN, AND $f(t)$ BE THE HEIGHT, WE HAVE

$$\begin{cases} f(0) = 299 \\ f(3.25) = 0. \end{cases} \qquad \left[3.25 = \frac{13}{4} \right]$$

SINCE $f(0) = 299$, $K = 299$. TO GET C, WE SOLVE

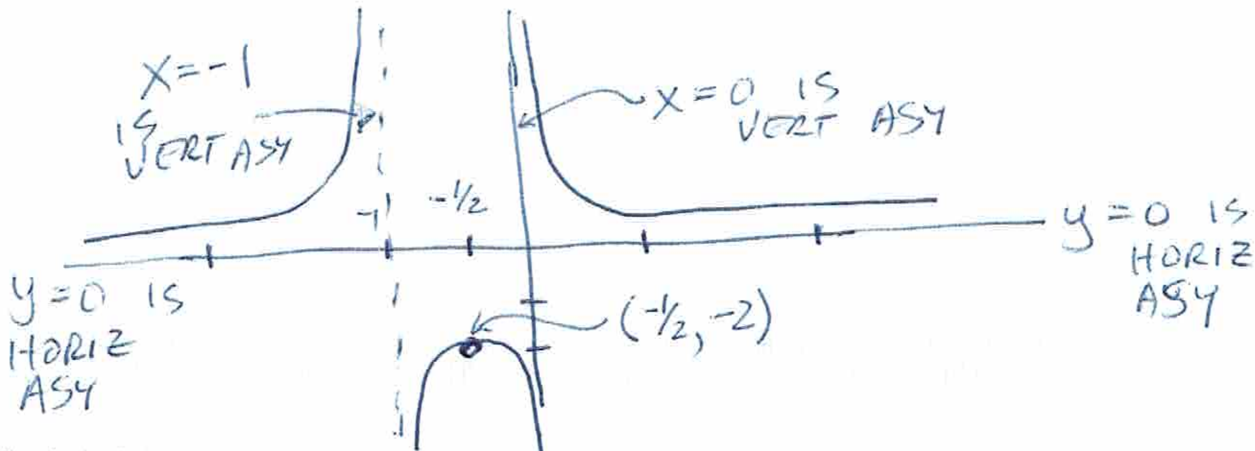
$$0 = f(3.25) = -16\left(\frac{13}{4}\right)^2 + C \cdot \frac{13}{4} + 299$$
$$= -169 + 299 + C \cdot 13/4 = 0$$

SO $\frac{13C}{4} = 130$, SO $C = 40$.

THIS MEANS

$$f'(t) = -32t + 40,$$
$$f'(0) = 40$$

SO THE BOY THREW THE ROCK <u>UP</u> AT 40 ft/SEC



5) Graph the following on a scaled set of axes. Label all critical points and any asymptotes. (Inflection points are not required.)

$$f(x) = y = \frac{1}{x+x^2} = (x+x^2)^{-1}$$

END BEHAVIOR:

$$\lim_{x \to \infty} \frac{1}{x+x^2} = 0,$$ SO $x = 0$ IS HORIZ ASYMPTOTE IN BOTH DIRS.

VERTICAL ASYMPTOTE/DOMAIN:

$$\frac{1}{x+x^2} = \frac{1}{x(x+1)},$$ SO VERT ASYMPT. AT $x = 0$, $x = 1$

CRITICAL POINTS:

BY CHAIN RULE,

$$f'(x) = -(x+x^2)^{-2}(1+2x).$$ $f'(x) = 0$ WHEN $1+2x = 0$, IE $\boxed{x = -1/2}$ IS ONLY CRIT PT

$$= -\frac{1+2x}{(x+x^2)^2}$$

$$f''(x) = -\frac{2(x+x^2)^2 - (1+2x)^2 \cdot 2(x+x^2)}{(x+x^2)^4}$$

$$f''\left(-\tfrac{1}{2}\right) = -\frac{2\left(-\frac{1}{2}+\frac{1}{4}\right)^2 - 0}{\left(-\frac{1}{2}+\frac{1}{4}\right)^4} < 0,$$ SO REL MAX AT $x = -1/2$, $y = -2$.

$$f\left(-\tfrac{1}{2}\right) = \frac{1}{-1/2+1/4} = -2$$

GRAPH IS ABOVE.