Stony Brook University

Dept. of Mathematics

MAT 125 – Calculus A

Final Exam

December 14, 2016

Please show all of your work.

PART 2

SOLUTIONS

9) Find the equation of the tangent line to $x^3 + 5x^2y - y^3 = 19$ at the point $(2,1)$

BY IMPLICIT DIFFERENTIATION,

$$3x^2 + 10xy + 5x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

AT (2,1) THIS MEANS

$$3\cdot(4) + 10\cdot 2\cdot 1 + 20\frac{dy}{dx} - 3\frac{dy}{dx} = 0$$

IE

$$32 + 17\frac{dy}{dx} = 0$$

SO $\frac{dy}{dx} = \frac{-32}{17}$

SO THE TANGENT LINE IS

Answer (8 points)

$$\boxed{y - 1 = -\frac{32}{17}(x-2)}$$

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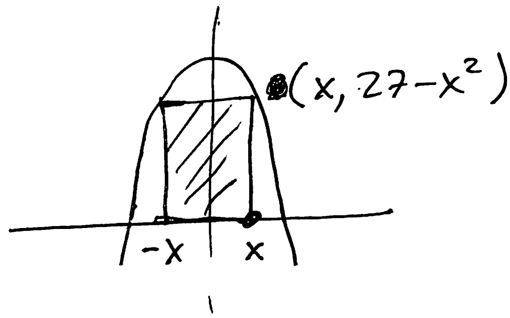
10) A rectangle is inscribed between the x-axis and the curve $y = 27 - x^2$, with its base on the x-axis and two vertices touching the curve. Find the area of the largest possible rectangle.



THE WIDTH OF THE RECTANGLE IS $2x$, THE HEIGHT IS $27 - x^2$.

SO AREA IS $A(x) = 2x(27 - x^2)$ WITH
$A(x) = 54x - 2x^3$. $0 < x < \sqrt{27}$.

FIND CRITICAL POINT,

$A'(x) = 54 - 6x^2$

$A'(x) = 0 \Leftrightarrow 54 = 6x^2$,

$9 = x^2$, SO $x = \pm 3$ BUT ONLY WANT $+3$.

MAXIMAL AREA IS

6×18 OR 108.

(IT IS A MAX SINCE $A''(x) = -12x$, $A''(3) < 0$ $\frown$ MAXIMUM)

Answer (10 points)

$6 \times 18 = 108$

Please show all of your work.

11) Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

FORM IS $\frac{0}{0}$,

BY L'HÔPITALS, $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}}{1} = \frac{1}{2}$

OR, IF YOU PREFER:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x\ (\sqrt{1+x}+1)} = \lim_{x\to 0}\frac{(1+x)-1}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

Answer (4 points)

$1/2$.

12) Find $f'(x)$ if $f(x)=\ln\left[\frac{(3x^2-5)^2}{(4x^2+5)^3}\right]$ $= 2\ln(3x^2-5) - 3\ln(4x^2+5)$

So $f'(x) = \frac{2}{3x^2-5}\cdot 6x - \frac{3}{4x^2+5}\cdot 8x = \frac{12x}{3x^2-5} - \frac{24x}{4x^2+5}$.

OR USE THE CHAIN RULE & QUOTIENT RULE:

$$\frac{(4x^2+5)^3}{(3x^2-5)^2}\cdot\left(\frac{2(3x^2-5)(4x^2+5)^3(6x) - (3x^2-5)^2(3)(4x^2+5)^2(8x)}{(4x^2+5)^6}\right)$$

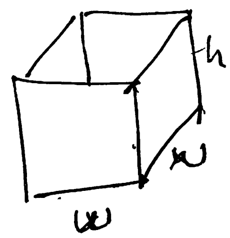
Answer (4 points)

THESE ARE SAME IF YOU SIMPLIFY.

Please show all of your work.

13) A rectangular box with a square base and no top is to be constructed so as to hold 32 in^3. Find the dimensions that minimize the surface area of the box. Justify that you have found the minimum and not the maximum.

LET THE BASE BE $w \times w$,
THE SIDES BE $w \times h$.



~~SA~~

SURFACE AREA IS $w^2 + 4wh$

BUT VOLUME $= w^2 h = 32$, SO $h = \frac{32}{w^2}$.

WANT TO MINIMIZE

$$S(w) = w^2 + \frac{128}{w}$$

$$S'(w) = 2w - \frac{128}{w^2} = \frac{2w^3 - 128}{w^2}$$

$S'(w) = 0$ WHEN $2w^3 = 128$, IE $w^3 = 64$
IE $w = 4$.

$S''(w) = 2 + 2 \cdot \frac{128}{w^3}$, $S''(4) > 0$, SO $w = 4$ IS A REL MIN.

~~DIM~~

IF $w = 4$, $h = \frac{32}{16} = 2$.

Answer (12 points)

DIMENSIONS ARE $4 \times 4 \times 2$

Please show all of your work

14) Find $\frac{d^2y}{dx^2}$ if $y^3 + y = x^2 - x$

BY IMPLICIT, $3y^2\frac{dy}{dx} + \frac{dy}{dx} = 2x-1$

SO $(3y^2+1)\frac{dy}{dx} = 2x-1$

$$\frac{dy}{dx} = \frac{2x-1}{3y^2+1}$$

IMPLICIT AGAIN:

$$\frac{d^2y}{dx^2} = \frac{2(3y^2+1) - (2x-1)\left(6y\frac{dy}{dx}\right)}{(3y^2+1)^2}$$

$$= \frac{6y^2+2 - (2x-1)(6y)(dy/dx)}{(3y^2+1)^2}$$

$$= \frac{6y+2 - 6y(2x-1)^2/(3y^2+1)}{(3y^2+1)^2}$$

Answer (8 points)

$$= \frac{(3y^2+1)(6y+2) - 6y(2x-1)^2}{(3y^2+1)^3}$$

Please show all of your work

15) The volume of a sphere is increasing at $24\pi\ cm^3/s$.

a) How fast is the radius increasing when the radius is 6 cm?

VOL OF SPHERE IS $\frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

THUS, WHEN $r=6$, WE HAVE

$$24\pi = 4\pi(36)\frac{dr}{dt}, \quad \text{SO} \quad \frac{24\pi}{4\cdot 36\pi} = \frac{dr}{dt}$$

SO $\frac{1}{6} = \frac{dr}{dt}$

Answer (5 points)

$$\boxed{\frac{dr}{dt} = 1/6 \text{ cm/sec}}$$

b) How fast is the surface area increasing at that time?

SURF. AREA OF SPHERE IS $4\pi r^2$

$$S = 4\pi r^2$$

SO

$$\frac{ds}{dt} = 8\pi r\frac{dr}{dt}$$

$$\frac{ds}{dt} = 8\pi(6)\left(\frac{1}{6}\right) = 8\pi$$

Answer (3 points)

$$\boxed{\frac{ds}{dt} = 8\pi \text{ cm}^2\text{/sec.}}$$

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16) Given $y = x^3 - 6x^2 - 36x + 4 := f(x)$

a) For what values of x is y decreasing?

b) What is the x-coordinate of the local minimum?

c) What is the x-coordinate of the point of inflection?

d) On the next page, sketch the curve, labeling what you believe is appropriate.

$f'(x) = 3x^2 - 12x - 36 = 3(x-6)(x+2)$

SO CRITICAL POINTS ARE $x=6$ AND $x=-2$.

$f''(x) = 6x - 12$.

SINCE $f''(-2) = -12-12 < 0$, -2 IS A LOCAL MAX

$f''(6) = 36 - 12 > 0$, SO 6 IS A LOCAL MIN

INFLECTION POINT ~~WHEN~~ WHEN $6x - 12 = 0$,

IE AT $x = 2$.

DECREASING WHEN $f'(x) < 0$, THAT IS, FOR

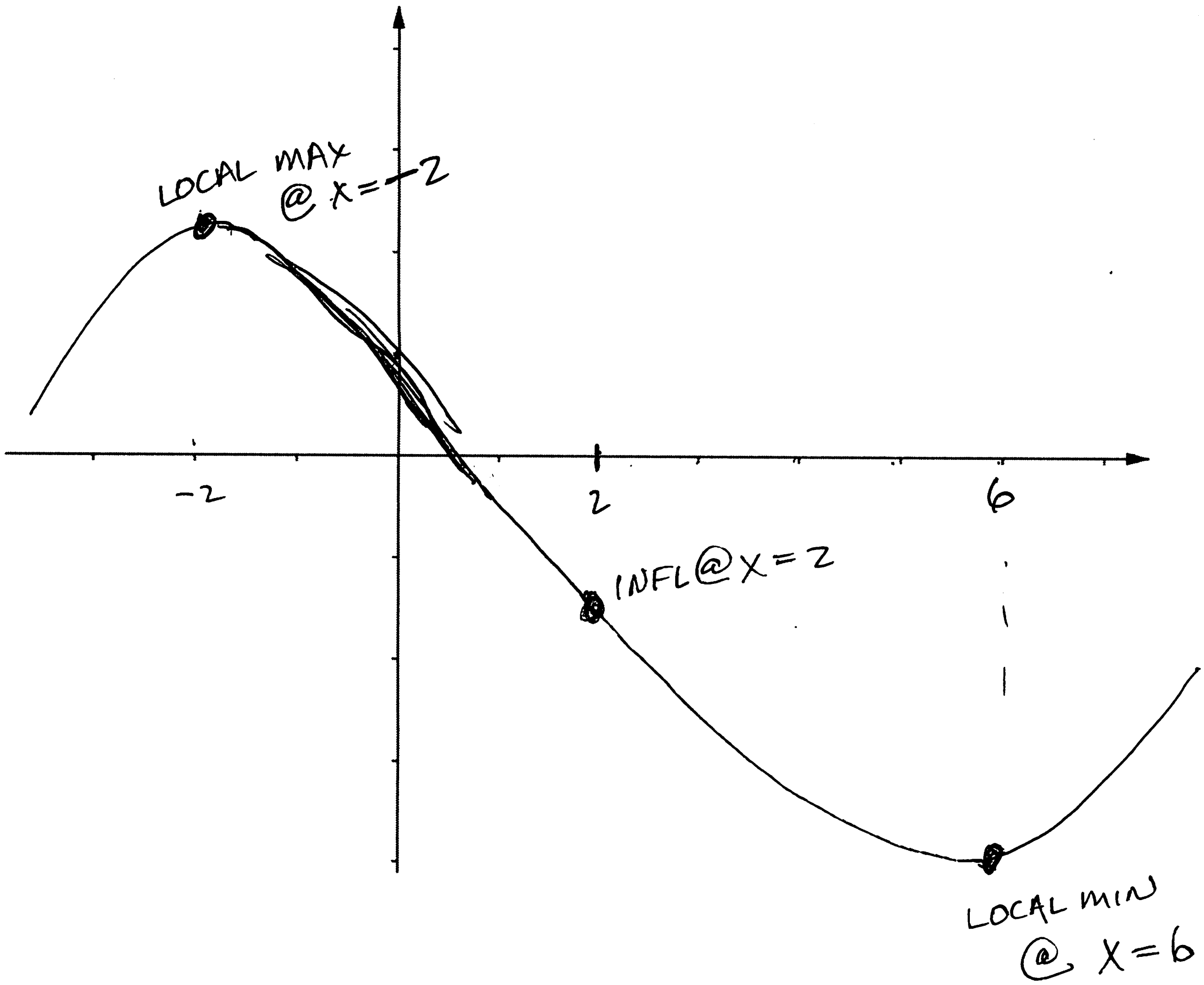
Answer (6 points) $-2 < x < 6$

a) $-2 < x < 6$

b) $x = -2$ IS LOCAL MAX

c) INFL @ $x = {}^{+}2$

d) Answer (4 points)



$f(-2) = \cancel{\ldots - 8 = 24 + 52 + 4 = 24}$

$f(2) = \cancel{8 - 12 \ldots}$

BLEAH NOT GONNA

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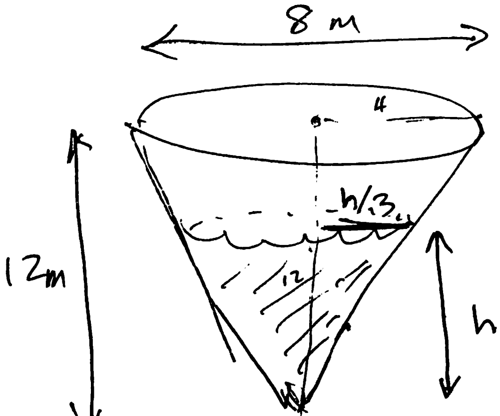
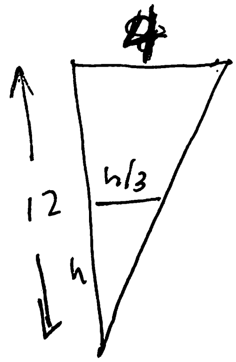
17) A conical tank is buried point down in the ground. Its diameter is 8 meters and its height is 24 meters. It is being filled with crude oil at $12\pi \ m^3 / hour$. How fast is the height of the oil in the tank rising when the height of the oil is 6 meters?





VOLUME OF CONE OF RADIUS r AND HEIGHT h IS

$$V = \frac{\pi}{3} r^2 h.$$

IN OUR CASE, $\frac{4}{12} = \frac{r}{h}$ BY SIMILAR TRIANGLES,

SO WHEN THE DEPTH IS h, THE RADIUS IS $h/3$.

SO $V = \frac{1}{27}\pi h^3$

$$\frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}.$$

FILLING AT $12\pi \ m^3/HR$

SO $\frac{dV}{dt} = 12\pi$

WANT $dh/dt @ h=6$

$$12\pi = \frac{\pi}{9}(36)\frac{dh}{dt}$$

$$\frac{12\pi \cdot 9}{36\pi} = 3 = \frac{dh}{dt}$$

Answer (12 points)

RISING AT 3 METERS / HOUR.

Extra credit:

Find $f^{(20)}(x)$ if $f(x) = x^{19}$.

$f'(x) = 19x^{18}$

$f^{(2)}(x) = 19 \cdot 18 x^{17}$

$f^{(3)}(x) = 19 \cdot 18 \cdot 17 x^{16}$

$\vdots$

$f^{(18)}(x) = 19 \cdot 18 \cdot 17 \cdot \cdots 3 \cdot 2 x$

$f^{(19)}(x) = 19 \cdot 18 \cdot 17 \cdot \cdots 2 \cdot 1$ A CONSTANT

So $f^{(20)}(x) = 0$.

Answer (4 points)

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