MAT 125 Solutions to Midterm 2 (Space)

1. Compute the derivatives for each of the functions below.

4 points (a) $f(x) = 6x^4 - 5x^{-2} + \sqrt{2x} - \pi^3$.

Solution: $f'(x) = 24x^3 + 10x^{-1} + \frac{\sqrt{2}}{2}x^{-1/2}$.
Don't forget that π^3 is just a number about equal to 30, so its derivative is zero.
A lot of people had trouble with the derivative of $\sqrt{2x}$; remember that $\sqrt{2x} = \sqrt{2}\sqrt{x} = \sqrt{2}x^{1/2}$, so the derivative is $\sqrt{2}(\frac{1}{2}x^{-1/2})$, or $\dfrac{1}{\sqrt{2x}}$. Or use the chain rule on $(2x)^{1/2}$.

4 points (b) $g(x) = x^5e^{3x}$

Solution: We need both the product rule and the chain rule (for the derivative of e^{3x}, which is $3e^{3x}$) here:

$$g'(x) = 5x^4e^{3x} + 3x^5e^{3x}.$$

4 points (c) $R(x) = \dfrac{x^3+12}{12-2x^3}$

Solution: By the quotient rule,

$$R'(x) = \frac{(3x^2)(12-2x^3) - (x^3+12)(-6x^2)}{(12-2x^3)^2} = \frac{36x^2+72x^2}{(12-2x^3)^2} = \frac{109x^2}{(12-2x^3)^2}.$$

Simplification is **not** necessary, so the first answer is fine.

2. Calculate the indicated derivatives.

4 points (a) $\dfrac{d}{dr}\left(\dfrac{4}{r} - \dfrac{r}{4}\right)$.

Solution: Don't be silly and use the quotient rule here. You want the derivative of $4r^{-1} - \frac{1}{4}r$, which is just $-4r^{-2} - \frac{1}{4}$.
If you use the quotient rule correctly, you should get an equivalent answer:

$$\frac{0\cdot r - 4\cdot 1}{r^2} - \frac{1\cdot 4 - r\cdot 0}{4^2} = -\frac{4}{r^2} - \frac{1}{4},$$

but that is a ton of work with many places to go wrong. Why work hard when you don't have to?

4 points

(b) $L(x) = \ln\left(\frac{x^2-4}{x^2+4}\right)$. Find $L'(1)$.

Solution: The easiest way to do this is to simplify first, then take the derivative. We have $L(x) = \ln(x^2-4) - \ln(x^2+4)$ so by the chain rule, we get $L'(x) = \frac{2x}{x^2-4} - \frac{2x}{x^2+4}$. If you prefer, you can use the chain rule and then the quotient rule to get

$$L'(x) = \frac{x^2+4}{x^2-4} \cdot \frac{2x(x^2+4) - 2x(x^2-4)}{(x^2+4)^2}.$$

Since we want $L'(1)$, we just plug 1 into either of the above to get $-\frac{2}{3} - \frac{2}{5} = \frac{-16}{15}$.

Although nobody noticed (not even the instructors), $L'(1)$ is in fact not defined, because $L(x)$ is only defined for $|x| > 2$. But we didn't take off for that.

4 points

(c) $\frac{d}{dx}\ln(\sec 3x)$

Solution: Using the chain rule we get $\frac{1}{\sec 3x} \cdot \sec(3x)\tan(3x) \cdot 3 = 3\tan(3x)$.

3. Let $f(x) = xe^{-3x}$.

5 points

(a) Calculate $f'(x)$.

Solution: Using the product rule (and then the chain rule for the derivative of e^{-3x}), we get $f'(x) = e^{-3x} - 3xe^{-3x}$.

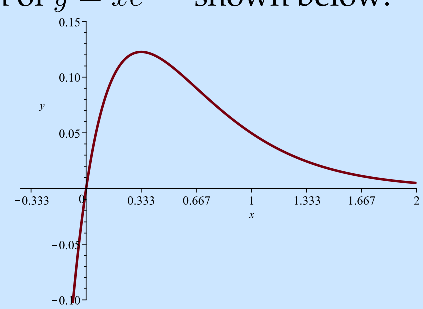
5 points

(b) For what values of x is $f(x)$ decreasing? If there are none, write "NONE"; otherwise, describe *all* such x.

Solution: The function will be decreasing when $f'(x) < 0$. From the first part, we have

$$f'(x) = e^{-3x} - 3xe^{-3x} = e^{-3x}(1-3x).$$

Since $e^{-3x} > 0$ for all x, we need x so that $1 - 3x < 0$, that is, $f(x)$ is decreasing when $x > \frac{1}{3}$. This is illustrated by the graph of $y = xe^{-3x}$ shown below:



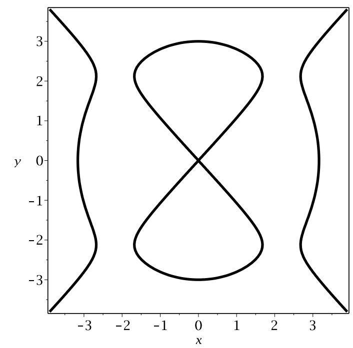
12 points

4. The set of points (x, y) which satisfy the relationship

$$y^2(y^2 - 9) = x^2(x^2 - 10)$$

lie on what is known as a "devil's curve", shown at right.

Write the equation of the line tangent to the given devil's curve at the point $(-\sqrt{10}, 3)$.



Solution:

First, we use implicit differentiation to determine the slope of the tangent line. This will be slightly easier if we rewrite the equation as $y^4 - 9y^2 = x^4 - 10x^2$ first. Differentiating with respect to x gives

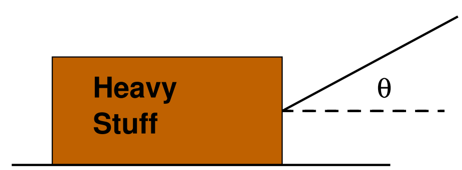
$$4y^3y' - 9 \cdot 2y \cdot y' = 4x^3 - 10 \cdot 2x \quad \text{and so} \quad y' = \frac{x(2x^2 - 10)}{y(2y^2 - 9)}.$$

At our desired point, $x = -\sqrt{10}$ and $y = 3$, and so the slope is $y' = \frac{-\sqrt{10} \cdot 10}{3 \cdot 9} = -\frac{10\sqrt{10}}{27}$.

This means the desired line is

$$y - 3 = -\frac{10\sqrt{10}}{27}(x + \sqrt{10}).$$

5. When an object with weight W is dragged horizontally by a rope attached to the object which makes an angle θ with the horizontal, the force required has magnitude



$$F(\theta) = \frac{\mu W}{\mu \sin\theta + \cos\theta},$$

where the number μ is called the *coefficient of friction*.

5 points

(a) If the object is a 40 lb box of rocks with a coefficient of friction μ of $1/2$, find $F'(\theta)$.

Solution: Plugging in $W = 40$ and $\mu = 1/2$, we get

$$F(\theta) = \frac{20}{\frac{1}{2}\sin\theta + \cos\theta} = 20(\tfrac{1}{2}\sin\theta + \cos\theta)^{-1},$$

so

$$F'(\theta) = -20(\tfrac{1}{2}\sin\theta + \cos\theta)^{-2}(\tfrac{1}{2}\cos\theta - \sin\theta) = -20\frac{\frac{1}{2}\cos\theta - \sin\theta}{(\frac{1}{2}\sin\theta + \cos\theta)^2}$$

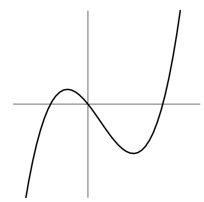
5 points

(b) Suppose you are pulling the box of rocks and the rope makes an angle of $\pi/6$ with the horizontal. Does it make sense to lower the rope (that is, decrease θ)? Explain why or why not (no points for a correct answer without a valid explanation).

Solution: Observe that $F'(\pi/6) = -20\frac{\sqrt{3}/4-1/2}{(1/4+\sqrt{3}/2)^2}$. In particular, since $\sqrt{3} < 2$, the numerator is negative, so $F'(\pi/6) > 0$. This means increasing the angle also increases the force needed to pull the box, and decreasing the angle decreases the force. So, yes, it makes sense to decrease the angle of the rope.

(The angle that minimizes the force is $\arctan(1/2)$, which is between $\pi/6$ and $\pi/7$ — so don't lower it too much. But you don't need to know that to do this problem.)

6. At right is the graph of **the derivative** $f'(x)$ of a function $f(x)$. Use it to answer each of the following questions.



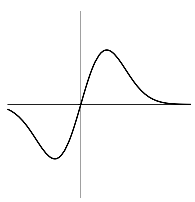
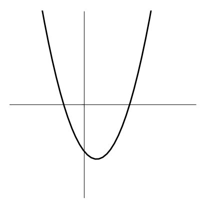
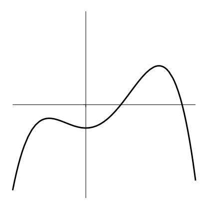
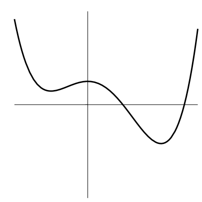
4 points

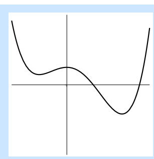
(a) Is $f(x)$ concave up, concave down, or neither at $x = 0$? Fully justify your answer.

Solution: Since the derivative is decreasing at $x = 0$, we know $f(x)$ is concave down there.

4 points

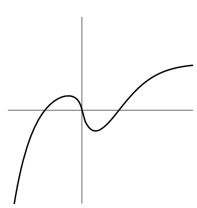
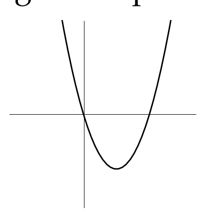
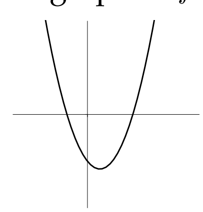
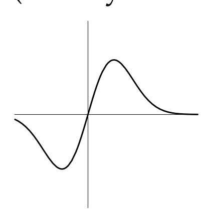
(b) Which of the following best represents the graph of $f(x)$? (circle your answer).

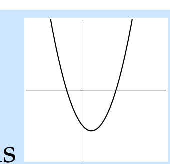





Solution: The graph of $f(x)$ is 

4 points

(c) Which of the following best represents the graph of $f''(x)$? (circle your answer).

Solution: The graph of $f''(x)$ is 

8 points 7. A function $H(x)$ measures the total "Hapyness" of a cube (yes, it is spelled that way—cubes don't spel so gud), depending on its width x. The Hapyness per unit volume is given by $Q(x) = \dfrac{H(x)}{x^3}$. If $H(10) = 4$ and $H'(10) = -6$, compute the rate of change of Hapyness per unit volume when $x = 10$, that is, find $Q'(10)$.

Solution: By the quotient rule,

$$Q'(x) = \frac{H'(x) \cdot x^3 - H(x) \cdot (3x^2)}{x^6}$$

and so $$Q'(10) = \frac{(-6)(100) - (4)(300)}{10^6} = -\frac{1800}{1000000} = -0.0018$$