## Math 125 Solutions to Midterm 1 (Viserion)

1. A tank holding 2100 gallons of maple syrup can be drained completely in three hours by opening a valve at its bottom. The amount of syrup in the tank at time t (where t is in hours, with  $0 \le t \le 3$ ) is given by

$$V(t) = 2100 \left(1 - \frac{t}{3}\right)^2 = \frac{2100}{9} (3 - t)^2.$$

5 points (a) Calculate the average rate (in gallons per hour) at which the syrup drains between t = 1 hour and t = 2 hours.

**Solution:** This is just the change in volume divided by the change in time. Since the change in time is 1 hour, we have

average rate 
$$=\frac{V(2)-V(1)}{2-1} = \frac{\frac{2100}{9} - \frac{2100}{9} \cdot 4}{1} = -2100/3 = -700 \text{ gal/hr}.$$

(A positive answer is also correct, depending on how you interpret the phrase "rate at which it is draining".)

(b) Write a formula that represents the average rate (in gallons per hour) at which the syrup drains between 2 hours some time *t* hours (with  $0 \le t \le 3$ ,  $t \ne 2$ ).

**Solution:** As above, we have

average rate 
$$= \frac{V(t) - V(2)}{t - 2} = \frac{2100}{9} \left( \frac{(3 - t)^2 - 1}{t - 2} \right) = \frac{2100}{9} \left( \frac{9 - 6t + t^2 - 1}{t - 2} \right)$$
$$= \frac{2100}{9} \left( \frac{8 - 6t + t^2}{t - 2} \right)$$
$$= \frac{2100}{9} \left( \frac{(t - 2)(t - 4)}{t - 2} \right)$$
$$= \frac{2100}{9} (t - 4).$$

(Depending on how you interpret it,  $\frac{2100}{9}(4-t)$  is also correct.)

5 points

(c) Calculate the instantaneous rate (in gallons per hour) at which the syrup is draining out at time t = 2 hours (depending on how you write it, this could be negative).

**Solution:** This is just the limit of the previous part as  $x \to 2$ . We have

$$\lim_{x \to 2} \frac{2100}{9}(t-4) = \frac{2100}{9}(-2) = -\frac{1400}{3} \frac{\text{gal}}{\text{hr}}$$

5 points

2. Compute each of the following limits. If the limit is not a finite number, please distinguish between  $+\infty$ ,  $-\infty$ , and a limit which does not exist (DNE). Justify your answer, at least a little bit.

5 points

(a)  $\lim_{x \to \pi/4} x^2 \tan x$ 

**Solution:** Since the function is continuous at  $\pi/4$ , we just evaluate:

$$\left(\frac{\pi}{4}\right)^2 \tan\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} \cdot 1 = \frac{\pi^2}{16} \,.$$

(b)  $\lim_{x \to 2} \frac{x^2 - 4}{6x^2 - 12x}$ 

Solution:

$$\lim_{x \to 2} \frac{(x-2)(x+2)}{6x(x-2)} = \lim_{x \to 2} \frac{(x+2)}{6x} = \frac{2+2}{12} = \frac{1}{3}.$$

(c) 
$$\lim_{x \to 2^+} \frac{x^3 - 4x^2 + 1}{(x - 5)(x - 2)}$$

**Solution:** Note for x close to 2, the numerator is close to -15 while the denominator tends towards zero. Thus, the function becomes unbounded at 2. Since we are looking at values of x > 2, the denominator is always negative. Hence, the limit is  $+\infty$ .

3. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between  $+\infty$ ,  $-\infty$ , and a limit which does not exist (DNE). Justify your answer, at least a little bit.

(a) 
$$\lim_{x \to -\infty} \frac{20x^5 - 1}{5x^5 - 27x^3 + 10}$$

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**Solution:** Since x is tending to infinity, the  $x^5$  term dominates the others, so we have

 $\lim_{x \to -\infty} \frac{20x^5 - 1}{5x^5 - 27x^3 + 10} = \lim_{x \to -\infty} \frac{20x^5}{5x^5} = \lim_{x \to -\infty} \frac{20}{5} = 4.$ 

5 points

(b)  $\lim_{x \to 5} \frac{x^2 - 25}{|x - 5|}$ 

**Solution:** Because of the absolute value, we need to consider x < 5 and x > 5 separately. For x > 5, |x - 5| = (x - 5), and we get

$$\lim_{x \to 5^+} \frac{x - 25}{|x - 5|} = \lim_{x \to 5^+} \frac{(x - 5)(x + 5)}{x - 5} = \lim_{x \to 5^+} x + 5 = 10.$$

When x < 5, |x - 5| = -(x - 5), yielding

$$\lim_{x \to 5^{-}} \frac{x - 25}{|x - 5|} = \lim_{x \to 5^{-}} -(x + 5) = -10.$$

Since the two one-sided limits differ, the limit does not exist.

2 points

7 points

(c)  $\lim_{x \to 4} \frac{\sqrt{x-2}}{x-4}$ 

**Solution:** We need to use the conjugate to deal with the square root:

$$\lim_{x \to 4} \frac{\sqrt{x-2}}{x-4} = \lim_{x \to 4} \frac{\sqrt{x-2}}{x-4} \left(\frac{\sqrt{x+2}}{\sqrt{x+2}}\right) = \lim_{x \to 4} \frac{x-4}{(x-4)(\sqrt{x+2})} = \lim_{x \to 4} \frac{1}{\sqrt{x+2}} = \frac{1}{6}.$$

4. For 
$$0 < x < 2$$
, let  $h(x) = \frac{3}{x} + \frac{3}{x-2} + \cos(\pi x)$ .

(a) What is h(1)? Simplify as much as possible, or write "Not Defined" if it is not defined.

**Solution:**  $h(1) = 3 - 3 + \cos(\pi) = -1$ .

5 points (b) Calculate  $\lim_{x \to 0^+} h(x)$  and  $\lim_{x \to 2^-} h(x)$ . If either limit is not a number, distinguish between  $+\infty$ ,  $-\infty$ , and DNE (does not exist).

**Solution:**  $\lim_{x\to 0^+} h(x) = +\infty$  (because for x small but positive, 1/x is extremely large and positive while  $-\frac{1}{x-2}$  is close to -1/2 and  $\cos(\pi x)$  is close to 1). Similarly,  $\lim_{x\to 2^-} h(x) = -\infty$ .

(c) Is there a value of x between 0 and 2 so that h(x) = 0? If so, is the value of x bigger than, equal to, or less than 1? If not, why not? How do you know? (Fully justify your answer.)

**Solution:** Since h(1) < 0 and  $\lim_{x\to 0^+} h(x) = +\infty$ , we know there are values of h that are positive for x close to 0. Since h(1) < 0 and h is continuous, there must be an x between 0 and 1 with h(x) = 0 by the Intermediate Value Theorem.

## 10 points 5. Explain in words what $\lim_{x \to 3} f(x) = +\infty$ means.

**Solution:** This just means that f(x) becomes arbitrarily large for values of x sufficiently close to 3.

Alternatively, we could say that f has a vertical asymptote at x = 3, but then we would also need to specify that f increases for x < 3 and decreases for x > 3 as well. This is necessary in order to distinguish between cases like  $\lim_{x\to 3} f(x) = -\infty$ , or when we have  $\lim_{x\to 3^+} f(x) = +\infty$  but  $\lim_{x\to 3^-} f(x) = -\infty$ , etc., which also have a vertical asymptote at x = 3, but  $\lim_{x\to 3} f(x) \neq +\infty$ .

20 points 6. The graph of a function f(x) is shown at right, for -4 < x < 4. Answer the questions below, assuming that for values not shown on the graph, the function continues in the same way (i.e., essentially straight). When calculating limits, distinguish between  $+\infty$ ,  $-\infty$ , and "does not exist".



(a) What is  $\lim_{x \to +\infty} f(x)$ , if it exists?

**Solution:** *f* has a horizontal asymptote on the right at y = 2, so  $\lim_{x \to +\infty} f(x) = 2$ .

(b) What is  $\lim_{x \to 2^{-}} f(x)$ , if it exists?

**Solution:** As *x* tends to 2 from below, the graph limits on y = 2, so  $\lim_{x \to 2^{-}} f(x) = 2$ .

(c) What is  $\lim_{x\to 2} \left[ (x-2)f(x) \right]$ , if it exists?

**Solution:** Note that for  $1 \le x \le 3$ , we have  $0 \le f(x) \le 3$ . This means that we also have  $0 \le (x-2)f(x) \le 3(x-2)$ . Using the Squeeze Theorem, we have

$$0 \le \lim_{x \to 2} \left[ (x-2)f(x) \right] \le \lim_{x \to 2} 3(x-2) = 0,$$

so the limit is 0.

(d) List all points x between -4 and 4 where f(x) is defined, but f(x) is not continuous.

**Solution:** f(x) is not continuous at x = -3, x = -1, x = 0, x = 1 and x = 2. However, it is also not defined when x is one of the points -3,-1, or 0, so the answer is only x = 1 and x = 2.

## 10 points 7. What value of *k* is necessary so that the function

$$f(x) = \begin{cases} kx + 2 + \ln 2 & x < 1\\ 2kx^2 + x & x \ge 1 \end{cases}$$

is continuous for all positive values of *x*? If there is no such value, write "NONE". Justify your answer fully.

**Solution:** The function f is continuous at all positive values of x except possibly at x = 1, since  $kx + 2 + \ln(2)$  and  $2kx^2 + x$  continuous all values of x. We just need to choose k so that the ends meet.

Since f(1) = 2k + 1, we need to find k so that  $\lim_{x \to 1^-} f(x) = 2k + 1$ . Observe that

 $\lim_{x \to 1^{-}} f(x) = \lim_{x \to 1^{-}} kx + 2 + \ln 2 = k + 2 + \ln 2,$ so we need to take k so that  $2k + 1 = k + 2 + \ln 2$ , that is,  $k = 2 - 1 + \ln(2) = 1 + \ln 2$ .

10 points 8. Write a limit that represents the slope of the line tangent to the graph of the function

$$f(x) = \begin{cases} (1-x)\tan(\frac{\pi}{2}x) & x \neq 1\\ \frac{2}{\pi} & x = 1 \end{cases}$$

at x = 1. You **do not need to evaluate the limit.** 

**Solution:** We just use the definition of the derivative at x = 1:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

Since *h* is not zero,  $f(1+h) = h \tan(\frac{\pi}{2}(1+h))$  and  $f(1) = 2/\pi$ . So,

$$f'(1) = \lim_{h \to 0} \frac{h \tan(\frac{\pi}{2}(1+h)) - 2/\pi}{h}$$

If you prefer to use the version of the definition with  $x \rightarrow 1$ , you would have

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(1 - x)\tan(\pi x/2) - 2/\pi}{x - 1}$$