MAT 125 Solutions to Midterm 1 (Rhaegal)

1. A tank holding 1500 gallons of maple syrup can be drained completely in three hours by opening a valve at its bottom. The amount of syrup in the tank at time t (where t is in hours, with $0 \le t \le 3$) is given by

$$V(t) = 1500\left(1 - \frac{t}{3}\right)^2 = \frac{1500}{9}(3-t)^2.$$

5 points

(a) Calculate the average rate (in gallons per hour) at which the syrup drains between $t = 1$ hour and $t = 2$ hours.

Solution: This is just the change in volume divided by the change in time. Since the change in time is 1 hour, we have

$$\text{average rate} = \frac{V(2) - V(1)}{2-1} = \frac{\frac{1500}{9} - \frac{1500}{9}\cdot 4}{1} = -1500/3 = -500 \text{ gal/hr}.$$

(A positive answer is also correct, depending on how you interpret the phrase "rate at which it is draining".)

5 points

(b) Write a formula that represents the average rate (in gallons per hour) at which the syrup drains between 2 hours some time t hours (with $0 \le t \le 3$, $t \ne 2$).

Solution: As above, we have

$$\begin{aligned}
\text{average rate} = \frac{V(t) - V(2)}{t-2} &= \frac{1500}{9}\left(\frac{(3-t)^2 - 1}{t-2}\right) = \frac{1500}{9}\left(\frac{9 - 6t + t^2 - 1}{t-2}\right) \\
&= \frac{1500}{9}\left(\frac{8 - 6t + t^2}{t-2}\right) \\
&= \frac{1500}{9}\left(\frac{(t-2)(t-4)}{t-2}\right) \\
&= \frac{1500}{9}(t-4).
\end{aligned}$$

(Depending on how you interpret it, $\frac{1500}{9}(4-t)$ is also correct.)

5 points

(c) Calculate the instantaneous rate (in gallons per hour) at which the syrup is draining out at time $t = 2$ hours (depending on how you write it, this could be negative).

Solution: This is just the limit of the previous part as $x \to 2$. We have

$$\lim_{x\to 2} \frac{1500}{9}(t-4) = \frac{1500}{9}(-2) = -\frac{1000}{3}\,\frac{\text{gal}}{\text{hr}}$$

2. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

5 points (a) $\lim_{x \to \pi/4} x^2 \tan x$

Solution: Since the function is continuous at $\pi/4$, we just evaluate:

$$\left(\frac{\pi}{4}\right)^2 \tan\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} \cdot 1 = \frac{\pi^2}{16}.$$

5 points (b) $\lim_{x \to 1} \frac{x^2 - 1}{7x^2 - 7x}$

Solution:

$$\lim_{x \to 1} \frac{(x-1)(x+1)}{7x(x-1)} = \lim_{x \to 1} \frac{(x+1)}{7x} = \frac{1+1}{7} = \frac{2}{7}.$$

5 points (c) $\lim_{x \to 1^+} \frac{x^3 - 4x^2 + 1}{(x-5)(x-1)}$

Solution: Note for x close to 1, the numerator is close to -2 while the denominator tends towards zero. Thus, the function becomes unbounded at 1. Since we are looking at values of $x > 1$, the denominator is always negative. Hence, the limit is $+\infty$.

3. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

5 points (a) $\lim_{x \to -\infty} \frac{8x^5 - 1}{4x^5 - 27x^3 + 10}$

Solution: Since x is tending to infinity, the x^5 term dominates the others, so we have

$$\lim_{x \to -\infty} \frac{8x^5 - 1}{4x^5 - 27x^3 + 10} = \lim_{x \to -\infty} \frac{8x^5}{4x^5} = \lim_{x \to -\infty} \frac{8}{4} = 2.$$

5 points (b) $\lim_{x\to 3} \frac{x^2-9}{|x-3|}$

Solution: Because of the absolute value, we need to consider $x < 3$ and $x > 3$ separately. For $x > 3$, $|x-3| = (x-3)$, and we get

$$\lim_{x\to 3^+} \frac{x-9}{|x-3|} = \lim_{x\to 3^+} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^+} x+3 = 6.$$

When $x < 3$, $|x-3| = -(x-3)$, yielding

$$\lim_{x\to 3^-} \frac{x-9}{|x-3|} = \lim_{x\to 3^-} -(x+3) = -6.$$

Since the two one-sided limits differ, the limit does not exist.

5 points (c) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

Solution: We need to use the conjugate to deal with the square root:

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right) = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{6}.$$

4. For $0 < x < 2$, let $h(x) = \frac{2}{x} + \frac{2}{x-2} + \cos(\pi x)$.

2 points (a) What is $h(1)$? Simplify as much as possible, or write "Not Defined" if it is not defined.

Solution: $h(1) = 2 - 2 + \cos(\pi) = -1$.

5 points (b) Calculate $\lim_{x\to 0^+} h(x)$ and $\lim_{x\to 2^-} h(x)$. If either limit is not a number, distinguish between $+\infty$, $-\infty$, and DNE (does not exist).

Solution: $\lim_{x\to 0^+} h(x) = +\infty$ (because for x small but positive, $1/x$ is extremely large and positive while $-\frac{1}{x-2}$ is close to $-1/2$ and $\cos(\pi x)$ is close to 1). Similarly, $\lim_{x\to 2^-} h(x) = -\infty$.

7 points (c) Is there a value of x between 0 and 2 so that $h(x) = 0$?
If so, is the value of x bigger than, equal to, or less than 1? If not, why not?
How do you know? (Fully justify your answer.)

Solution: Since $h(1) < 0$ and $\lim_{x\to 0^+} h(x) = +\infty$, we know there are values of h that are positive for x close to 0. Since $h(1) < 0$ and h is continuous, there must be an x between 0 and 1 with $h(x) = 0$ by the Intermediate Value Theorem.

10 points

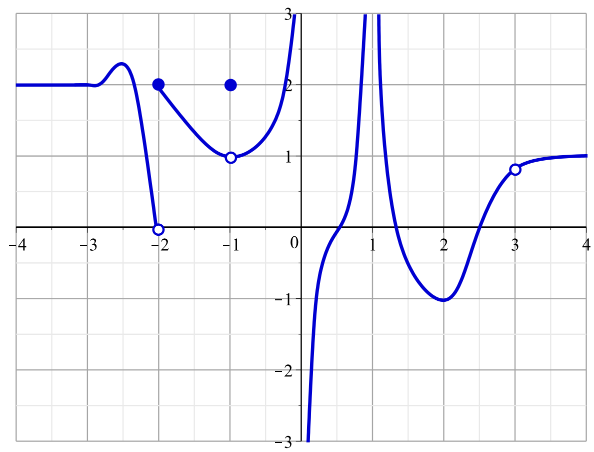
5. Explain in words what $\lim_{x\to 3} f(x) = +\infty$ means.

Solution: This just means that $f(x)$ becomes arbitrarily large for values of x sufficiently close to 3.

Alternatively, we could say that f has a vertical asymptote at $x = 3$, but then we would also need to specify that f increases for $x < 3$ and decreases for $x > 3$ as well. This is necessary in order to distinguish between cases like $\lim_{x\to 3} f(x) = -\infty$, or when we have $\lim_{x\to 3^+} f(x) = +\infty$ but $\lim_{x\to 3^-} f(x) = -\infty$, etc., which also have a vertical asymptote at $x = 3$, but $\lim_{x\to 3} f(x) \neq +\infty$.

20 points

6. The graph of a function $f(x)$ is shown at right, for $-4 < x < 4$. Answer the questions below, assuming that for values not shown on the graph, the function continues in the same way (i.e., essentially straight).
When calculating limits, distinguish between $+\infty$, $-\infty$, and "does not exist".



(a) What is $\lim_{x\to +\infty} f(x)$, if it exists?

Solution: f has a horizontal asymptote on the right at $y = 1$, so $\lim_{x\to +\infty} f(x) = 1$.

(b) What is $\lim_{x\to\ -2^+} f(x)$, if it exists?

Solution: As x tends to -2 from above, the graph limits on $y = 2$, so $\lim_{x\to\ -2^+} f(x) = 2$.

(c) What is $\lim_{x\to -2} \Big[(x+2)f(x)\Big]$, if it exists?

Solution: Note that for $-3 \le x \le -1$, we have $0 \le f(x) \le 3$. This means that we also have $0 \le (x+2)f(x) \le 3(x+2)$. Using the Squeeze Theorem, we have

$$0 \le \lim_{x\to -2} \Big[(x+2)f(x)\Big] \le \lim_{x\to -2} 3(x+2) = 0,$$

so the limit is 0.

(d) List all points x between -4 and 4 where $f(x)$ is defined, but $f(x)$ is not continuous.

Solution: $f(x)$ is not continuous at $x = -2$, $x = -1$, $x = 0$, $x = 1$ and $x = 3$. However, it is also not defined when x is one of the points 0,1, or 3, so the answer is only $x = -2$ and $x = -1$.

10 points 7. What value of k is necessary so that the function

$$f(x) = \begin{cases} kx + 3 + \ln 2 & x < 1 \\ 2kx^2 + x & x \geq 1 \end{cases}$$

is continuous for all positive values of x? If there is no such value, write "NONE". Justify your answer fully.

Solution: The function f is continuous at all positive values of x except possibly at $x = 1$, since $kx + 3 + \ln(2)$ and $2kx^2 + x$ continuous all values of x. We just need to choose k so that the ends meet.

Since $f(1) = 2k + 1$, we need to find k so that $\lim_{x\to 1^-} f(x) = 2k + 1$.

Observe that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} kx + 3 + \ln 2 = k + 3 + \ln 2,$$

so we need to take k so that $2k + 1 = k + 3 + \ln 2$, that is, $k = 3 - 1 + \ln(2) = 3 + \ln 2$.

10 points 8. Write a limit that represents the slope of the line tangent to the graph of the function

$$f(x) = \begin{cases} (1 - x)\tan(\frac{\pi}{2}x) & x \neq 1 \\ \frac{2}{\pi} & x = 1 \end{cases}$$

at $x = 1$. You **do not need to evaluate the limit.**

Solution: We just use the definition of the derivative at $x = 1$:

$$f'(1) = \lim_{h\to 0} \frac{f(1 + h) - f(1)}{h}.$$

Since h is not zero, $f(1 + h) = h\tan(\frac{\pi}{2}(1 + h))$ and $f(1) = 2/\pi$. So,

$$f'(1) = \lim_{h\to 0} \frac{h\tan(\frac{\pi}{2}(1 + h)) - 2/\pi}{h}.$$

If you prefer to use the version of the definition with $x \to 1$, you would have

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{(1 - x)\tan(\pi x/2) - 2/\pi}{x - 1}.$$